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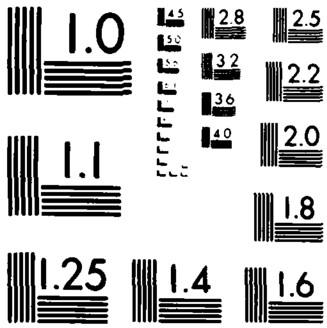

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ON MAXIMIZING THE EXPECTED LIFETIME OF REPLACEABLE SYSTEMS

by

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Chapter 1

Introduction

A major concern in reliability is minimizing or delaying the effect of failures of critical equipment. Two examples of this concern in practice are the design of equipment with redundant components and the scheduling of possibly expensive inspections of equipment in service. This paper will concentrate on the scheduled maintenance of functioning equipment.

The general optimal maintenance problem involves determining a schedule according to which inspections, repairs, and replacements are made which optimizes some objective. The simplest models have involved single state, single component systems, a failure cost, a replacement cost, no inspections or repairs and a long-run cost minimizing objective. The aim was to try to replace the component just before it failed, balancing the cost of replacement against the cost of failure. See Barlow and Proschan [1965] for some discussion of these simple models. More recent optimal maintenance models involve multiple states, multiple components and intricate cost structures. The objective is still to minimize long-run costs. Sherif and Smith [1981] give a good review of the optimal maintenance literature.

In practice, many of these newer models suffer from two problems. One problem is that normally, many costs and probabilities need to be specified in order to use the model. Even worse, in many cases no reasonable algorithm is given for determining the optimal replacement schedule, rather it is only determined that this optimal policy has some property. This paper deals with these two shortcomings by considering a less intricate model which would still have wide application.

Basic Model: A system consists of a single component so that the system fails when the component fails. A finite number of spares are available for the component. Replacements are made in zero time and no costs are associated with failure or replacement. Once replaced, a component cannot be reused. The only objective in the Basic Model is to maximize the expected time until the first failure of an installed component by setting an optimal replacement schedule. The time until the first installed component failure is called the system lifetime.

In designing the replacement schedule, one attempts to balance the possibility and expense of failure against a limited costly supply of spare components. The best application would be a system with a single expensive, vital component for which failure is catastrophic, but replacement is possible. For example, the design of replacement schedules for heart pace-makers or the engine of a single-engine airplane. Another application would be in a hierarchical approach to system reliability, where this model would be used for overall system planning and more detailed models would be used for developing

replacement schedules for specific components of the system. Our model is relatively easy to use in practice because no costs need to be specified and the optimal replacement policy is easily computable under a variety of assumptions.

In defining the Basic Model, we have not discussed how to model the lifetimes of individual spares. Two different modelling techniques give rise to two different models. These two models are then combined into a third model.

Complete Information Model: This model assumes that the common lifetime distribution of the spares is known. Further, we will assume that this distribution has a continuous density and a finite mean. This is the most difficult model to apply, since we are assuming more information here than in the other models; but it also gives the most accurate results.

Minimax Model: Rather than knowing the complete lifetime distribution of the spares, we only know that each lifetime distribution has certain properties, or equivalently, we know that each distribution is a member of a given class of lifetime distributions. An example, which will be used throughout this paper, is the class of all lifetime distributions with a given percentile. Under the Minimax Model, the objective is to specify a schedule which maximizes the expected system lifetime under the worst possible lifetime distributions. These spares will be known as minimax spares, and an optimal schedule will be known as a minimax schedule.

Mixed Model: In this model, the spares may each have different properties. Each spare may have a completely known lifetime distribution or may be a minimax spare. Among the spares with known lifetime distributions, each may have a different distribution, and the minimax spares may have different properties. The interesting aspect of this model is that, in addition to specifying the schedule, the order in which the spares are to be used must also be determined.

Derman et al. [1984] examined the Complete Information Model and their results are presented in Chapter 2. Their focus is on the properties of schedules and expected lifetimes for spares with identical and known lifetime distributions. The primary focus of this thesis will be the relaxation of the assumptions that the spare lifetime distribution is known, and that the spares are identical; i.e. the Minimax and Mixed Models. In Chapter 3, the minimax replacement time for a single spare is calculated under a variety of distributional assumptions. The most important of these assumptions is that the mean and variance of the lifetime are known. The results obtained in Chapter 3 are then employed in Chapter 4 to derive minimax schedules and their properties for multiple, identical spares. The Mixed Model, specifically the problem of determining the sequence in which different spares should be used, is discussed in Chapter 5. The problem is treated in a general context, for which sequencing spares is a special case. Chapter 6 discusses two variations of the Complete Information Model. In the first case, groups of spares may be used in parallel.

This means that a group of spares may be assembled into a super-spare which does not fail until each spare in the group has failed. Two important questions are what size groups to use and then how to sequence these groups. In the second case, an additional spare is received at some time in the future. Chapter 7 summarizes the results obtained and suggests some directions for further research.

Chapter 2

Complete Information Results

Derman, Lieberman, and Ross [1984] are the only authors to have considered the basic model of this paper. They treat the Complete Information Model described in Chapter 1. Specifically, n spares are available for a single component system. Replacements are scheduled in an attempt to maximize the expected system lifetime. Replaced components cannot be reused.

<u>Definitions and Assumptions</u>

Derman et al. assume the lifetimes of the spares to be independent and identically distributed with a known distribution function F. Let $\bar{F}(x) = 1-F(x)$. Further, they assume that F has a continuous density, f, concentrated on $(0,T)$, with $0 < T \leq \infty$, and

$$\mu = \int_0^T \bar{F}(x)dx < \infty.$$

Let $r(x) = f(x)/\bar{F}(x)$ be the failure rate of F.

Now we will be more precise in the definition of the objective. Let S be a replacement schedule, and L the time before the first failure

of the system with the replacement schedule S. The objective is to choose the replacement schedule so that $E_S L$, the expected failure time under the replacement schedule S, is maximized. Let v_n be the maximum expected lifetime of a system with n spares. If only one spare is available for the system, then there is only one schedule, the one that never replaces, and $v_1 = \mu$, the expected lifetime of the spare. For $n > 1$, v_n satisfies the recursion:

$$v_n = \max_{0 \leq t \leq T} \phi_n(t)$$

where

$$\phi_n(x) = \int_0^t xf(x)dx + \bar{F}(t)(t+v_{n-1})$$

$$= \int_0^t \bar{F}(x)dx + \bar{F}(t)v_{n-1} .$$

Let t_n be the value of t which maximizes $\phi_n(t)$ above. This value may not be unique, but at least one maximizing value does exist. Thus, t_n is an optimal replacement time for the first of n spares. If $t_n = 0$, then the first spare is not used and $v_n = v_{n-1}$. If $t_n = \infty$, then only the first spare is used and $v_n = \mu$. The collection $\{t_n\}_{1 \leq n \leq k}$ is called an optimal schedule, and v_k is called the value of the optimal schedule, or the optimal expected lifetime.

Derman et al. prove various properties of optimal schedules and provide examples of optimal schedules for some lifetime distributions. These results will be summarized shortly, but first we make an

observation about ϕ_n. Notice that $\phi_n(t)$ can also be viewed as a function of v_{n-1}, which will henceforth be known as the reward. The reward is viewed as the expected lifetime to be received after the successful replacement of the current component. In general, we can see that the reward for successful substitution could be any nonnegative real number, say v. Then we can define:

$$h_F(v) = \max_{0 \le t \le T} L_v(t,F), \text{ and}$$

$$t_F(v) = \operatorname{argmax}_{0 \le t \le T} L_v(t,F)$$

where

$$L_v(t,F) = \begin{cases} \int_0^t \bar{F}(x)dx + v\bar{F}(t), & 0 < t \le \infty \\ v, & 0 = t. \end{cases}$$

The special definition in the case $t = 0$ is required if $\bar{F}(0) < 1$, since when we let $t = 0$, we mean that we do not use this spare. If we let $v_1 = \mu$, and for $n > 1$ let $v_n = h_F(v_{n-1})$, $t_n = t_F(v_{n-1})$ we recover an optimal schedule and its associated optimal return. The function h will be known as the single spare return function, since it tells us how well we can do with this one spare, followed by an arbitrary reward. In later chapters, h will have different subscripts denoting different types of spares. Thus, for example, h_F denotes the return function for a single spare with known distribution function F. Similarly, $t_F(v)$ will be known as the single spare replacement time and will have different subscripts for different types of spares.

Results

Now we turn to the results obtained by Derman et al., some of which are recast using the notation introduced above. These results are presented without proof, but the proofs are available in Derman et al. The first result shows us when n spares are no better than one spare.

Proposition 2.1: $v_n = v_1$ for every $n > 1$ if and only if

$$\mu\bar{F}(t) \leq \int_t^T \bar{F}(x)dx, \quad 0 \leq t \leq T . \square$$

This condition is known as New Worse Than Used in Expectation (NWUE). All results which follow in this chapter will assume that F is not NWUE, i.e. $v_2 > v_1$. In this case, it follows that:

Proposition 2.2: $v_{n+1} > v_n$, $v_{n+2} - v_{n+1} < v_{n+1} - v_n$, and

$$0 < t_{n+1} < t_n < T, \quad n = 1, 2, 3, \ldots . \square$$

The optimal expected lifetime is strictly increasing in the number of spares, but its first difference is strictly decreasing. Further, any optimal schedule increases the time between replacements as the number of spares decreases.

The next result helps in determining the optimal schedule.

Proposition 2.3: For an optimal replacement time, t_n, $r(t_n) = 1/v_{n-1}$ and r cannot be decreasing at t_n. $\square$

If F has a strictly increasing failure rate (IFR), then Proposition 2.3 determines t_n uniquely. If r is unimodal and has two roots to $r(t) = 1/v_{n-1}$, then t_n is the smaller of the two roots. If r is "bathtub shaped," then t_n is the larger of the two roots.

The next result determines when the optimal lifetime is bounded as the number of spares approaches infinity.

Proposition 2.4: $\lim_{n\to\infty} v_n < \infty$ if and only if $f(0) > 0$. $\square$

Although t_n is not determined uniquely in general, $t^* = \lim_{n\to\infty} t_n$ is determined uniquely. The following proposition gives a sufficient condition for $t^* = 0$.

Proposition 2.5: If $\bar{F}(t+x) < \bar{F}(t)\bar{F}(x)$ for all $t,x > 0$ then $t^* = 0$. $\square$

The above stated condition is also known as New Better Than Used, Strictly (NBU(S)). This is a weaker condition than IFR or IFRA.

Derman et al. also consider optimal periodic schedules, where replacements occur after equal amounts of time. In this case, a schedule is determined by one number, the inter-replacement time. Two results from periodic schedules are of particular interest to us here. Let y_n be the optimal schedule with n spares, and let u_n be the optimal expected system lifetime.

Proposition 2.6: $\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n$, and

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} y_n. \square$$

Thus periodic and nonperiodic schedules have the same limiting properties, both in expected lifetime and in the actual scheduling of the replacements.

We will now give a computational example from Derman et al.

Example 2.1: Let $f(t) = 2t$, $0 \leq t \leq 1$. Then $F(t) = t^2$, $\mu = 2/3$ and $r(t) = 2t/(1-t^2)$. Since $f(0) = 0$, $\lim_{n\to\infty} v_n = \infty$. Further, since F is strictly IFR, it follows that $t^* = 0$ and the equality $r(t_n) = 1/v_{n-1}$ determines t_n uniquely. In the notation given above,

$$t_F(v) = \sqrt{v^2 + 1} - v, \text{ and}$$

$$h_F(v) = \mu\left(\sqrt{(v^2+1)^3} - v^3\right) .$$

Thus, v_n and t_n may be calculated recursively according to the formulas, $v_{n+1} = h_F(v_n)$ and $t_{n+1} = t_F(v_n)$, with the initial condition that $v_0 = 0$. Some actual values will be given at the end of Chapter 4, along with similar values for other models. □

This concludes the exposition of the results of Derman et al. Note that no general closed form solution for the v_n and t_n is given. However, if F is of one of the types described above (IFR, unimodal, "bathtub shaped"), then it is quite easy to find v_n and t_n. A more critical problem with the Complete Information Model is that the lifetime distribution of the spares must be known. Since it is our intent that the Basic Model be easy to implement, we will relax the

assumption that the complete lifetime distribution be known and move to the Minimax Model.

Chapter 3

Single Spare Minimax Results

In the previous chapter, we explored how to solve, at least in principle, the problem: $\max_t L_v(t,F)$. Suppose, however, that we do not know the actual distribution, F, of the spare lifetimes, but we do know that F lies in some class of distribution functions, D. Then we define $h_D(v) = \sup_t \inf_{F \in D} L_v(t,F)$ as the minimax solution for a single spare, or the minimax single spare return function. In Chapter 4, the full Minimax Model will be considered, with multiple spares.

The idea of the minimax approach is that we try to do the best that we can assuming the worst outcome of the information which we do not know. This approach is commonly used in game theory, and in a class of optimal inspection problems. Note that, assuming the supremum and infimum are attained, $h_D(v) = L_v(t^*,F^*) \leq L_v(t^*,F^o)$, where F^o is any distribution in D. Thus, if F^o is the actual lifetime distribution, then the predicted expected lifetime is a lower bound on the actual expected lifetime, given we use the schedule, t^*.

Before going on to prove the main single spare, minimax results, we will define the class of functions, H. This class of functions contains all single spare return functions, both for completely known and partially known lifetime distributions.

Definition 3.1: $h:\mathbb{R}^+ \to \mathbb{R}^+$, is a member of H provided:

1. $h(x) \geq x$, $\quad x \geq 0$,
2. $h(x) \leq h(y)$, $\quad y \geq x \geq 0$, and
3. $h(x) - x \geq h(y) - y$, $\quad y \geq x \geq 0$. □

Thus, a function in H lies above the line, $y=x$, is nondecreasing, but is always approaching the line, $y=x$. Henceforth, we will use the following notations:

$$\mu = h(0),$$

$$\bar{\mu} = \sup\{x \mid h(x) = \mu\}, \text{ and}$$

$$\bar{v} = \inf\{x \mid h(x) = x\}.$$

It is possible that $\bar{\mu} = 0$ and/or $\bar{v} = \infty$. But, $\bar{\mu} \leq \mu \leq \bar{v}$ since $h(x) = x$ implies that $x \geq \mu$, but $h(x) = \mu$ implies that $x \leq \mu$. We will now give one example of the usefulness of H. Suppose, $v_1 = \mu$ and $v_{n+1} = h(v_n)$ for $h \in H$ and $n = 1, 2, 3, \ldots$. Then $v_{n+1} \geq v_n$, $v_{n+1} - v_n \leq v_n - v_{n-1}$ for $n = 2, 3, 4, \ldots$ and $\lim_{n\to\infty} v_n = \bar{v}$. These first two results are just the first two results of Proposition 2.2, adjusted to allow NWUE distributions, and this third result is analogous to Proposition 2.4, and will be proven in Chapter 4.

Before proving that all single spare return functions belong to H, we will define the class of lifetime distributions from which all distributions used in this thesis will be chosen.

Definition 3.2: $C = \{\text{distribution functions, } F \mid F(0^-) = 0\}$. □

That is, C is the class of all lifetime distribution functions.

The next order of business is to prove that the single spare return functions do actually belong to H.

Theorem 3.1: h_F and h_D belong to H for all $F \epsilon C$ and $D \subset C$.

Proof: First we will consider h_D, taking the conditions one at a time.

1. $L_v(0,F) = v$ for all F, so $h_D(v) \geq v$.

2. If $u \geq v$ then $L_u(t,F) \geq L_v(t,F)$ for all t and F, so $h_D(u) \geq h_D(v)$.

3. If $u \geq v$ then $L_u(t,F)-u \leq L_v(t,F)-v$ for all t and F, so $h_D(u)-u \leq h_D(v)-v$.

Now if we let $D = \{F\}$, then we can see that $h_F = h_D$ and the theorem is proved. □

We will now consider the case of distribution functions with a known percentile as a simple application of the minimax concept. Suppose that $D = \{F \epsilon C \mid F(s^-) = p\}$. That is, D is the set of all lifetime distributions which have their p^{th} percentile at time s. Then the minimax return function is quite easy to compute.

Proposition 3.1: $\sup_t \inf_{F \epsilon D} L_v(t,F) = h_{p,s}(v)$, where

$$h_{p,s}(v) = \begin{cases} (s+v)(1-p), & 0 \leq v < s(1-p)/p, \\ v, & v \geq s(1-p)/p, \end{cases}$$

and

$$t_{p,s}(v) = \begin{cases} s^-, & 0 \leq v < s(1-p)/p, \\ 0, & v \geq s(1-p)/p. \end{cases}$$

__Proof__: No matter what value we choose for t, the distribution, F^*, minimizes $L_v(t,F)$, where:

$$F^*(t) = \begin{cases} 0, & t < 0, \\ p, & 0 \leq t < s, \\ 1, & s \leq t. \end{cases}$$

Clearly, there is no advantage in choosing t strictly between 0 and s. Thus, we either choose μ, v or let t approach s from below. A simple calculation gives the stated value for $h_{p,s}(v)$, including the fact that $t_{p,s}(v) = 0$ when $v \geq s(1-p)/p$. □

Thus, $\mu_{p,s} = s(1-p)$, $\bar{\mu}_{p,s} = 0$, and $\bar{v}_{p,s} = s(1-p)/p$ for distributions with their p^{th} percentile at s.

__Known Mean and Variance__

The most interesting minimax case for our model is the case in which we know the mean, μ, and the variance, σ^2, of the spare lifetime. Let $C_{\mu,\sigma}$ be the set of all lifetime distributions with mean, μ, and standard deviation, σ. More precisely:

$$C_{\mu,\sigma} = \{F \epsilon C \mid \int_0^\infty x dF(x) = \mu, \int_0^\infty x^2 dF(x) = \sigma^2 + \mu^2 \}.$$

__Theorem 3.2__: $\sup_t \inf_{F \epsilon C_{\mu,\sigma}} L_v(t,F) = h_{\mu,\sigma}(v) = \max\{\mu, v, \sup_{0 \leq t \leq \mu} B(t)\}$,

$$\text{and} \quad t_{\mu,\sigma}(v) = \begin{cases} 0 & , h_{\mu,\sigma}(v) = v, \\ \underset{0 \le t \le \mu}{\operatorname{argsup}} B(t) & , h_{\mu,\sigma}(v) > \max\{\mu,v\}, \\ \infty & , h_{\mu,\sigma}(v) = \mu, \end{cases}$$

$$\text{where} \quad B(t) = t + \frac{v(\mu-t)^2}{\sigma^2 + (\mu-t)^2} .$$

This theorem gives the single spare return function for a spare with known mean and variance. No explicit solution for $\sup_t B(t)$ is given. However, it turns out that this problem is quite easy to solve. It will be seen in Theorem 3.3 that any one-dimensional search may be used to solve $\sup_t B(t)$. Further, the maximizing value of t is the optimal replacement time for this spare with reward v.

The proof of Theorem 3.2 will require three lemmas which will be described briefly here, then proven, and finally combined into the complete proof of the theorem. The first lemma will show that we can reduce the class of distributions, $C_{\mu,\sigma}$, to $C^3_{\mu,\sigma}$; the class of "three-point" lifetime distributions with known mean and variance. An n-point distribution is a discrete distribution with n points of increase. The second lemma will attack the problem: $\inf_{F \in C^3_{\mu,\sigma}} L_v(t,F)$. Finally, the third lemma refines the result of the second lemma, to reach the desired result.

<u>Lemma 3.1</u>: $\inf_{F \in C_{\mu,\sigma}} L_v(t,F) = \inf_{F \in C^3_{\mu,\sigma}} L_v(t,F)$.

<u>Proof</u>: This follows from a result due to Wassilv Hoeffding [1955]. The following definitions will be used in this proof:

$$d(F,G) \equiv \sup_{-\infty<x<\infty} |F(x) - G(x)|,$$

$$C^{\infty}_{\mu,\sigma} \equiv \bigcup_{n=1}^{\infty} C^{n}_{\mu,\sigma}.$$

Thus, $d(F,G)$ is a norm in the space of distribution functions, and $C^{\infty}_{\mu,\sigma}$ is the class of all discrete distributions with given mean and variance.

We must check the following four conditions:

1. There must exist $K_{t,v}(x)$ so that $L_v(t,F) = \int K_{t,v}(x)dF(x)$.

$$K_{t,v}(x) = \begin{cases} x, & x \leq t, \\ t + v, & x > t. \end{cases}$$

2. $C_{\mu,\sigma} = \{F \epsilon C \mid \int g_i(x)dF(x) = c_i,\ i = 1, 2\}$.

$g_1(x) = x, \quad c_1 = \mu, \quad g_2(x) = x^2, \quad c_2 = \mu^2 + \sigma^2.$

3. $L_v(t,F)$ must be continuous in F under the norm d.

$d(F,G) < \delta$, implies that

$$|L_v(t,F) - L_v(t,G)| \leq |\int_0^t (\bar{F}(x)-\bar{G}(x))dx + v(\bar{F}(t)-\bar{G}(t))|$$

$$\leq (t + v)\delta.$$

Thus, $L_v(t,F)$ is continuous in F.

4. For every $F \epsilon C_{\mu,\sigma}$ and $\delta > 0$, there must exist $F^* \epsilon C^{\infty}_{\mu,\sigma}$ so that $d(F,F^*) < \delta$.

Hoeffding showed that this condition is satisfied in the case $g_i(x) = x^i$. □

Before going on to prove the next lemma, we will pause for a look at the properties of two- and three-point distributions with given mean and variance. These results are all given in Huang [1983], and are restated here without proof.

<u>Definition 3.3</u>: An n-point distribution will be denoted by $(y_1, \ldots, y_n)$, the n points of increase. The probability of the i^{th} outcome, y_i, is p_i. □

For a three-point distribution, $(y_1, y_2, y_3) \in C^3_{\mu,\sigma}$, the p_i's are determined as follows. This follows from solving: $\Sigma p_i = 1$, $\Sigma p_i y_i = \mu$, and $\Sigma p_i y_i^2 = \alpha\mu$.

$$p_1 = \frac{(y_3 - \mu)(y_2 - \beta_{y_3})}{(y_3 - y_1)(y_2 - y_1)}$$

$$= \frac{(\mu - y_2)(\beta_{y_2} - y_3)}{(y_3 - y_1)(y_2 - y_1)}$$

$$p_2 = \frac{(y_3 - \mu)(\beta_{y_3} - y_1)}{(y_3 - y_2)(y_2 - y_1)}$$

$$= \frac{(\mu - y_1)(y_3 - \beta_{y_1})}{(y_3 - y_2)(y_2 - y_1)}$$

$$p_3 = \frac{(\mu - y_2)(\beta_{y_2} - y_1)}{(y_3 - y_2)(y_3 - y_1)}$$

$$= \frac{(\mu - y_1)(\beta_{y_1} - y_2)}{(y_3 - y_2)(y_3 - y_1)}$$

where

$$\beta_y = \frac{\mu(\alpha - y)}{\mu - y},$$

and

$$\alpha = \frac{\mu^2 + \sigma^2}{\mu}.$$

In order for a three-point distribution to have mean, μ and variance, σ^2, the y_i's must satisfy a set of inequalities.

Proposition 3.2: For $(y_1, y_2, y_3) \in C^3_{\mu,\sigma}$, the following inequalities must hold:

$$0 \le y_1 < \beta_{y_3} < \mu < \alpha \le \beta_{y_1} < y_3 < \infty,$$

$$\beta_{y_3} < y_2 < \beta_{y_1},$$

$$\text{if } y_2 < \mu \text{ then } y_3 < \beta_{y_2}, \text{ and}$$

$$\text{if } y_2 > \mu \text{ then } y_1 > \beta_{y_2}.$$

In the case of a two-point distribution, the situation is much simpler. Given $(y_1, y_2) \in C^2_{\mu,\sigma}$, p_1 and p_2 are given by:

$$p_1 = \frac{y_2 - \mu}{y_2 - y_1} = \frac{\sigma^2}{\sigma^2 + (\mu - y_1)^2}, \quad p_2 = \frac{\mu - y_1}{y_2 - y_1}.$$

In addition, $y_2 = \beta_{y_1}$, $y_1 = \beta_{y_2}$, and

$$0 \le y_1 < \mu < \alpha \le y_2 < \infty.$$

We now have the necessary information on two- and three-point distributions, for the statement and proof of the second lemma.

Lemma 3.2: $\inf_{F \in C^3} L_v(t,F) = \inf_{0 \leq y \leq t} A(y)$, where

$$A(y) = t + v - \frac{\sigma^2(v + t - y)}{(\mu - y)^2 + \sigma^2},$$

provided,

$$\mu - t \in \frac{v}{2} \pm \sqrt{\frac{v^2}{4} - \sigma^2} \qquad (*).$$

Note that (*) requires $v \geq 2\sigma$.

Proof: Given that F is the three-point distribution (y_1, y_2, y_3):

$$L_v(t,F) = \begin{cases} t+v, & t < y_1, \\ p_1 y_1 + (1-p_1)(t+v), & y_1 \leq t < y_2, \\ p_1 y_1 + p_2 y_2 + p_3(t+v), & y_2 \leq t < y_3, \\ \mu, & y_3 \leq t. \end{cases}$$

Since we may choose t to guarantee $h_{\mu,\sigma}(v) = \max\{\mu, v\}$, we are not interested in how much $\inf_F L_v(t,F)$ can be reduced below μ or v. Rather, we are only interested in values of t which ensure $\inf_F L_v(t,F) > \max\{\mu, v\}$. There are four cases which need to be analyzed.

Case 1: $t < y_1$

Since $t+v \geq p_1 y_1 + (1-p_1)(t+v)$ if and only if $t+v \geq y_1$, and we can never guarantee $t < y_1$ with our choice of t, case 1 is dominated by case 2. That is, the distribution, F, will always "choose" $y_1 \leq t$.

Case 2: $y_1 \leq t < y_2$

In this case, $L_v(t,F) = p_1 y_1 + (1-p_1)(t+v) = t+v - p_1(t-y_1+v)$, recalling that p_1 is a function of y_1, y_2, and y_3. Since $\partial p_1/\partial y_2 \geq 0$ and $t+v \geq y_1$, it follows that $\partial L/\partial y_2 \leq 0$, so we must increase y_2 as much as possible. This leads to $y_2 = \beta y_1$, forcing $p_3 = 0$. Thus, we drop y_3 and are left the two-point distribution (y_1, y_2) with $y_1 \leq t < y_2$, and $p_1 = \sigma^2/(\sigma^2+(\mu-y_1)^2)$. Substituting p_1 into $L_v(t,F)$, given above, yields the function $A(y_1)$.

Case 3: $y_2 \leq t < y_3$

In this case, $L_v(t,F) = p_1 y_1 + p_2 y_2 + p_3(t+v) = \mu + p_3(t+v-y_3)$. Thus we must choose t to ensure that $t+v-y_3 \geq 0$. The only upper bound on y_3 applies when $y_2 < \mu$, so we must choose $t < \mu$. In this case, $y_3 < \beta y_2 < \beta_t$, so we impose the condition $\beta_t < t+v$ to guarantee that $\inf_F L_v(t,F) \geq \mu$. The condition $\beta_t < t+v$ is equivalent to (*). Assuming (*) holds, it is easy to see that $\partial L/\partial y_3 < 0$, so we increase y_3 as much as possible, which again leaves us with a two-point distribution as $y_1 \to 0$. Since the two-point distribution is (y_2, y_3), and $y_2 \leq t < y_3$, the situation is the same as in case 2.

Case 4: $y_3 \leq t$

It cannot be the case that $y_3 \leq t$, since we have already seen that we must choose $t \leq \mu$, while $y_3 > \mu$.

Thus, case 1 and case 4 will never actually happen, and cases 2 and 3 give the same value for $\inf_F L_v(t,F)$, assuming (*). We have also seen that if (*) is false, then $\inf_F L_v(t,F) = \max(\mu, v)$. □

Lemma 3.3: (*) implies that:

$$L_v(t,F) = t + \frac{v(\mu - t)^2}{\sigma^2 + (\mu - t)^2} = A(t).$$

Proof: In order to prove this, we will see that $A'(y)$ is negative on the interval $[0,t]$. Through fairly simple algebra it can be seen that:

$$A'(y) = \left[\frac{-(\mu-y)^2 - 2(v+t-\mu)(\mu-y) + \sigma^2}{(\mu-y)^2 + \sigma^2}\right]\sigma^2.$$

The sign of $A'(y)$ depends only on the sign of its numerator, $-(\mu-y)^2 - 2(v+t-\mu)(\mu-y) + \sigma^2 \equiv a(v)$, since the denominator of $A'(y)$ is nonnegative for all feasible values of y. The function $a(y)$ is simply a parabola in $(\mu-y)$ with maximum at $y = v+t$. Thus, $a(y)$ is decreasing over the interval $[0,t] \subset [0,t+v]$. All that remains to be seen is that $A'(t) < 0$, which would imply that $A'(y) < 0$ for all y in the interval $[0,t]$. In order to determine the sign of $A'(t)$, we need only determine the sign of $a(t)$.

If we consider $a(t)$ as a function of t, it can be shown that if t satisfies (*), then $a(t) < 0$. Since $a(t)$ is a parabola opening upwards, if $a(t_1) < 0$ and $a(t_2) < 0$ then $a(t) < 0$ for all $t \in [t_1,t_2]$. By evaluating $a(t)$ at the endpoints of the interval in (*), we find that $a(t) < 0$ when (*) holds. Thus, $A'(y) < 0$ for $y \in [0,t]$, so $A(y)$ is minimized by choosing $y = t$. $\square$

We are now in a position to give a complete proof of Theorem 3.2.

Theorem 3.2: $\sup_t \inf_{F\in C_{\mu,\sigma}} L_v(t,F) = h_{\mu,\sigma}(v) = \max\{\mu, v, \sup_{0\leq t\leq\mu} B(t)\}$,

$$\text{and} \quad t_{\mu,\sigma}(v) = \begin{cases} 0 & , h_{\mu,\sigma}(v) = v, \\ \underset{0\leq t\leq\mu}{\operatorname{argsup}} B(t) & , h_{\mu,\sigma}(v) > \max\{\mu,v\}, \\ \infty & , h_{\mu,\sigma}(v) = \mu, \end{cases}$$

$$\text{where} \quad B(t) = t + \frac{v(\mu-t)^2}{\sigma^2 + (\mu-t)^2}.$$

Proof: By choosing $t=0$ or $t=\infty$, we can guarantee $h_{\mu,\sigma}(v) = \mu$ or v, respectively. Lemmas 3.1 through 3.3 prove that if $h_{\mu,\sigma}(v) > \max\{\mu,v\}$, then $h_{\mu,\sigma}(v) = \sup_t B(t)$ as stated. □

The next theorem gives some details of the solution of the problem: $\sup_t B(t)$.

Theorem 3.3: $dB(t)/dt$ is strictly monotone decreasing on $[0,\mu-\sigma/\sqrt{3}]$. If $B'(t^*) = 0$ for $t^* \in [0,\mu-\sigma/\sqrt{3}]$ then $\sup_t B(t) = \max\{B(0),B(\mu),B(t^*)\}$. If $B'(t)$ has no zero for $t \in [0,\mu-\sigma/\sqrt{3}]$, then $\sup_t B(t) = \max\{B(0),B(\mu)\}$.

Proof:

$$B'(t) = 1 - \frac{2v\sigma^2(\mu - t)}{(\sigma^2 + (\mu-t)^2)^2} \quad , \text{ and}$$

$$B''(t) = \frac{2v\sigma^2(\sigma^2 - 3(\mu-t)^2)}{(\sigma^2 + (\mu-t)^2)^3}.$$

Thus, $B''(t) <(>) 0$ when $\sigma^2 <(>) 3(\mu-t)^2$. A relative maximum, if one exists must occur in the interval $[0,\mu-\sigma/\sqrt{3}]$, because that is where the second derivative is negative. Since $B''(t)$ is negative on $(0,\mu-\sigma/\sqrt{3})$, $B'(t)$ must be strictly monotone on this interval. $\square$

Bounds and Parameters

Now that we have seen how to compute $t_{\mu,\sigma}$ and $v_{\mu,\sigma}$, we will give the equations for $\bar{\mu}_{\mu,\sigma}$, and $\bar{v}_{\mu,\sigma}$.

Theorem 3.4: If $\mu \geq 2\sigma$ then $\bar{\mu}_{\mu,\sigma} = 2\sigma$ and

$$\bar{v}_{\mu,\sigma} = \frac{\mu}{27}\left[2\left(\frac{\mu}{\sigma}\right)^2 + 18 + \left(2\left(\frac{\mu}{\sigma}\right)^2 - 6\right)\sqrt{1 - 3/\left(\frac{\mu}{\sigma}\right)^2}\right],$$

otherwise, $\bar{\mu}_{\mu,\sigma} = \bar{v}_{\mu,\sigma} = \mu$.

Proof: If $v < 2\sigma$, then (*) cannot hold and $h_{\mu,\sigma}(v) = \max\{\mu,v\}$. Thus, $\bar{\mu}_{\mu,\sigma} = \bar{v}_{\mu,\sigma} = \mu$, since for $v < \mu$, $h_{\mu,\sigma}(v) = \mu$, and for $v > \mu$, $h_{\mu,\sigma}(v) = v$. The fact that $\bar{\mu}_{\mu,\sigma} = 2\sigma$ follows from the fact that (*) requires $v \geq 2\sigma$, and if $\mu > v > 2\sigma$ then we can guarantee $h_{\mu,\sigma}(v) > \mu = \max\{\mu,v\}$ by choosing $t = \mu - \sigma$.

To get $\bar{v}_{\mu,\sigma}$, we find the largest v so that, for some t, $v = B(t)$. Simple algebra shows that

$$v \leq t + \frac{v(\mu-t)^2}{\sigma^2 + (\mu-t)^2} \quad \text{if and only if}$$

$$v\sigma^2 \leq t^3 - 2\mu t^2 + (\sigma^2 + \mu^2)t \equiv g(t).$$

By solving $g'(t) = 0$ and checking the second derivative, we can see that $g(t)$ is maximized at:

$$t^o = \frac{\mu}{3}\left(2 - \left(1 - 3\frac{\sigma^2}{\mu^2}\right)^{1/2}\right).$$

Now, $\bar{v}_{\mu,\sigma} = g(t^o)/\sigma^2$, which, after a significant amount of algebra, yields the stated result. □

This proof gives the following useful corollary.

Corollary 3.1: $t_{\mu,\sigma}(\bar{v}_{\mu,\sigma}) = t^o$.

Proof: t^o solves the equation $B(t) = \bar{v}_{\mu,\sigma}$, for $v = \bar{v}_{\mu,\sigma}$, so it must be an optimal t for this value of v. □

The following theorem summarizes some bounds on $t_{\mu,\sigma}$ which are useful, both theoretically and in the actual computation of $h_{\mu,\sigma}$, since the bounds reduce the size of the interval over which we must search for the optimal t.

Theorem 3.5: If $v \in (\bar{\mu}_{\mu,\sigma}, \bar{v}_{\mu,\sigma})$ then

$$\frac{\sigma^2}{\mu^2} v \leq t_{\mu,\sigma}(v) \leq \mu - \sigma, \quad t_{\mu,\sigma}(v) \geq x - v, \text{ and } \quad t_{\mu,\sigma} \geq t^o.$$

Proof: Choose $(0,x)$ as a two point distribution. then $p_2 = \mu/x$ $= \mu^2/(\mu^2 + \sigma^2)$ and $L_v(t,F) = p_2(t+v)$ for all the reasonable F's. In order to assure $p_2(t+v) \geq \mu$, we get $t \geq x - v$ (which may be negative). The inequality, $p_2(t+v) \geq v$, guarantees that $t \geq v\sigma^2/\mu^2$.

To see that $t_{\mu,\sigma} \leq \mu - \sigma$, we first notice that $t_{\mu,\sigma}(2\sigma) = \mu - \sigma$. This may be checked by substituting $v = 2\sigma$ and $t = \mu - \sigma$ into the expression for $B'(t)$ given in Theorem 3.3. Then we use the fact, to be proven in the next chapter, that $t_{\mu,\sigma}(v)$ decreases as v increases. The final inequality also follows from this fact, along with Corollary 3.1. □

This completes the single spare minimax results. The most important result being the derivation of the minimax schedule and expected lifetime for distributions with known mean and variance, given in Theorem 3.2. This and the other results will be the basis for the following chapters.

Chapter 4

Identical Spares

In this chapter, we will consider the problem of scheduling multiple, identical spares, under both the Minimax and Complete Information Models. The focus will be on restating the results of Derman et al., as given in Chapter 2, for the Minimax Model. This chapter will be presented in three parts. The first will consider the case of a finite number of spares. The next section will be concerned with the calculation of limiting values as the number of spares approaches infinity. The final section will be an example of computation and a comparison of schedules and expected lifetimes under a variety of assumptions on the amount of information known.

Finite Spares

Before considering the Minimax Model with known mean and variance, we will return to the example of distributions with known percentile. Once again, let $D = \{F \epsilon C \mid F(s^-) = p\}$. For the remainder of this thesis, we will assume $0 < p < 1$ and $0 < s < \infty$. The values for $h_{p,s}(v)$ and $t_{p,s}(v)$ are given in Chapter 3. The transition from a single spare return function to an optimal schedule for multiple,

identical spares is actually an easy one. Let $v_0 = 0$, $v_n = h_{p,s}(v_{n-1})$, and $t_n = t_{p,s}(v_{n-1})$, $n = 1, 2, 3, \ldots$. Now v_n is the minimax expected lifetime with n spares available, and t_n is the minimax replacement time of the n^{th} spare. As might be expected, v_n and t_n may be calculated explicitly in the case of a known percentile.

Proposition 4.1: Given v_n and t_n as defined above,

$$v_n = s(1-(1-p)^n)(1-p)/p, \quad n = 1, 2, 3, \ldots, \text{ and}$$
$$t_n = s^-, \quad n = 1, 2, 3, \ldots .$$

Proof: This proof will be by induction. Note that $v_1 = s(1-p)$, and assume $v_n = s(1-(1-p)^n)(1-p)/p$. Then

$$v_{n+1} = \left(s(1-(1-p)^n)\frac{(1-p)}{p} + s\right)(1-p) = s(1-(1-p)^{n+1})\frac{(1-p)}{p} .$$

Since $v_{n-1} < s(1-p)/p$, $t_n = s^-$ for $n \geq 1$. □

Proposition 4.2: Given v_n and t_n as defined above,

$$v_{n+1} > v_n, \quad v_{n+2} - v_{n+1} < v_{n+1} - v_n, \quad \text{and } t_{n+1} = t_n, \; n \geq 1.$$

Proof: Using the facts that $0 < p < 1$, and $0 < s < \infty$ and substituting the values from Proposition 4.1 yields the desired results. □

One of the themes of this section will be monotonicity results for v_n and t_n, not only as n varies, but as the parameters defining the class of distributions change. Thus, at this time, we are interested in the way v_n and t_n behave as p and s are varied.

Proposition 4.3: v_n is increasing in s and decreasing in p, while t_n is increasing in s, but is unchanged by changes in p.

Proof: That v_n and t_n are increasing in s is obvious from Proposition 4.1. It is also clear that t_n does not change with p. To see that v_n is decreasing in p, note that $dv_n/dp < 0$ if and only if $(1-p)^{-n} > (1+np)$. But

$$(1-p)^{-n} = (1 + p + p^2 + \dots)^n > 1 + np.$$

Thus the final assertion is proved. $\square$

Now that we have seen the ease with which a known percentile may be handled, we will move to the case of a known mean and variance. Once again we define $v_0 = 0$, $v_{n+1} = h_{\mu,\sigma}(v_n)$, and $t_{n+1} = t_{\mu,\sigma}(v_n)$, $n \geq 0$. In this case we give no closed-form expression for v_n in terms of μ and σ^2. However, as was shown in the previous chapter, the recursive calculation of the v_n's and t_n's is not hard. Even without an explicit expression for v_n or t_n we may still recover the interesting monotonicity results given for the other models.

We are primarily interested in how v_n and t_n vary with μ, σ^2, and n, but we will prove slightly more general results in the next theorem.

Theorem 4.1: If $h_{\mu,\sigma}(v) > \max\{\mu,v\}$, then,

1. $t_{\mu,\sigma}$ is increasing in μ,
2. $\mu - t_{\mu,\sigma}$ is constant in μ,
3. $t_{\mu,\sigma}$ is decreasing in v,

4. $t_{\mu,\sigma} + v$ is increasing in v,

5. $t_{\mu,\sigma}$ is decreasing in σ,

6. $t_{\mu,\sigma} + \sigma$ is increasing in σ,

7. $h_{\mu,\sigma}$ is increasing in μ, and

8. $h_{\mu,\sigma}$ is decreasing in σ.

Proof: To prove the first two assertions, note that μ and t only appear in $B'(t)$ in the form $(\mu-t)$. Thus, if v and σ^2 are unchanged, then $(\mu-t)$ must be constant to insure that $B'(t) = 0$.

The third and fourth assertions are proved as follows. The equation, $B'(t) = 0$, is equivalent to the equation,

$$v = \frac{(\sigma^2 + (\mu - t)^2)^2}{2\sigma^2(\mu - t)} .$$

We can now compute dv/dt and see that $dv/dt < -1$ if and only if $\sigma^4 < 3(\mu-t)^4$. But this follows from $t \leq \mu - \sigma$, which is guaranteed by Theorem 3.5. This proves both assertions.

The next assertion to prove is that t is decreasing in σ^2. The equation $B'(t) = 0$ is equivalent to the equation,

$$(\sigma^2 + (\mu-t)^2)^2 = 2v\sigma^2(\mu-t).$$

Considering t as an implicit function of σ and differentiating, we find:

$$\frac{dt}{d\sigma} = \frac{4v\sigma(\mu-t) - 4\sigma(\sigma^2+(\mu-t)^2)}{2v\sigma^2 - 4(\mu-t)(\sigma^2 + (\mu-t)^2)} .$$

Using the fact that $\mu - t \geq \sigma$, as proven in Theorem 3.5, and substituting v out with $B'(t) = 0$, it can be seen that the numerator of this fraction is positive and the denominator is negative. The proof of the sixth assertion uses the same equation for $dt/d\sigma$, once again substituting out for v, to see that $dt/d\sigma > -1$.

The proofs of the last two assertions are quite easy. Considering $B(t)$ as a function of σ, we can see that, for fixed t, $B(t)$ decreases as σ increases. Thus, it must be that $\sup_t B(t)$ also decreases as σ increases. Similarly, when $B(t)$ is considered as a function of μ, it can be seen that for fixed t, $B(t)$ increases as μ increases. $\square$

This lemma is necessary for the proof of the next theorem. We already know that $0 \leq dh_{\mu,\sigma}/dv \leq 1$ since $h_{\mu,\sigma} \in H$. Now we will also see that $h_{\mu,\sigma}$ is convex.

<u>Lemma 4.1</u>: $dh_{\mu,\sigma}/dv$ is nondecreasing in v.

<u>Proof</u>: For $v \in (\bar{\mu}_{\mu,\sigma}, \bar{v}_{\mu,\sigma})$, let $t(v)$ be the minimax replacement time considered as a function of v. Then $h_{\mu,\sigma}(v) = \dot{B}(t(v)) \equiv \bar{B}(v,t(v))$. Now $dB/dv = \partial\bar{B}/\partial v + (\partial\bar{B}/\partial t)(dt(v)/dv)$, but one condition defining $t(v)$ is that $\partial\bar{B}/\partial t$, evaluated at $t(v)$, is equal to 0. So, dB/dv, evaluated at $t(v)$, is equal to $\partial\bar{B}/\partial v$, evaluated at $t(v)$. That is,

$$\frac{dh_{\mu,\sigma}(v)}{dv} = \frac{(\mu - t(v))^2}{\sigma^2 + (\mu - t(v))^2} .$$

But this is increasing in v since $t(v)$ is decreasing in v, and σ^2 is positive. When $v < \bar{\mu}$, $h'_{\mu,\sigma}(v) = 0$, and when $v > \bar{v}$, $h'_{\mu,\sigma}(v) = 1$, which completes the proof that $h'_{\mu,\sigma}$ is nondecreasing in v. $\square$

We are now in position to prove the monotonicity results for the v_n's and t_n's. These results will be contained in the next two theorems, the first of which deals with the variation of n, and the second of which deals with the variation of μ and σ^2.

<u>Theorem 4.2</u>: Given v_n and t_n as defined above, if $v_2 > v_1$ then $v_{n+1} > v_n$, $v_{n+2} - v_{n+1} < v_{n+1} - v_n$, and $t_{n+1} < t_n$, $n \geq 1$.

<u>Proof</u>: Since $h_{\mu,\sigma} \in H$, we know $v_{n+1} \geq v_n$, and $v_{n+2} - v_{n+1} \leq v_{n+1} - v_n$, $n \geq 1$. All that remains to be seen is that the inequalities are strict. Suppose $v_{n+1} = v_n > v_{n-1}$. Then $h_{\mu,\sigma}(v_{n-1}) = v_n$ and $h_{\mu,\sigma}(v_{n-1}+\Delta) = v_n$, where $\Delta = v_n - v_{n-1} > 0$. Thus by the Mean Value Theorem, $h'_{\mu,\sigma}(v) = 0$ for some $v \in [v_{n-1}, v_n]$. But this contradicts Lemma 4.1 since $v_2 = h_{\mu,\sigma}(v_1) > v_1$.

Similarly, if $v_{n+2} - v_{n+1} = v_{n+1} - v_n$, then $h_{\mu,\sigma}(v_{n+1}) - h_{\mu,\sigma}(v_n) = v_{n+1} - v_n$ implies, by the Mean Value Theorem, that $h'_{\mu,\sigma}(v) = 1$ for some $v \in [v_{n+1}, v_n]$, but this is a contradiction since $v_{n+1} < \bar{v}_{\mu,\sigma}$ and $h'_{\mu,\sigma}$ is increasing.

The fact that $t_{n+1} < t_n$ follows from the facts that $v_n > v_{n-1}$ and that $t_{\mu,\sigma}$ is strictly decreasing in v. $\square$

This theorem is the same statement as Proposition 2.2, applied to minimax schedules with known mean and variance. Notice that the proof uses only the facts that $h_{\mu,\sigma} \in H$, $h_{\mu,\sigma}$ is convex and that $t_{\mu,\sigma}$ is strictly decreasing in v for $v \in (\bar{\mu}_{\mu,\sigma}, \bar{v}_{\mu,\sigma})$. Thus, parts or all of this theorem can probably be proved in a more general setting. The next theorem gives monotonicity results as μ and σ vary. No analogous result exists in Derman et al.

<u>Theorem 4.3</u>: If $v_2 > v_1$ then v_n increases with μ and decreases with σ.

<u>Proof</u>: By induction, $v_1 = \mu$ which is increasing in μ. Assuming v_n has increased with the increase in μ, $v_{n+1} = h_{\mu,\sigma}(v_n)$ must also increase since $h_{\mu,\sigma}(v)$ increases with v and μ. The proof for changing σ is similar. $\square$

Thus, v_n behaves as we would expect. When the spares have a larger mean and/or a smaller variance, the minimax expected lifetime is longer.

The situation with t_n is somewhat complicated by the fact that, since $t_n = t_{\mu,\sigma}(v_n)$, t_n is affected by the change in μ or σ as well as the change in v_n. It is not the case that t_n is monotone in σ, as we can see from the following counterexample.

Consider two spares, both with mean equal to 1. If the first spare has a variance of .1, while the second spare has a variance of .24, then t_2 is larger for the first spare, but t_4 is larger for the second

spare. The actual numbers are: for the first spare, $t_2 = .55$, $t_4 = .47$; while for the second spare, $t_2 = .50$ and $t_4 = .49$. The question of whether t_n is monotone in μ is still open.

Once the optimal or minimax schedule has been computed, it is possible to calculate the cumulative distribution function of the system lifetime, T_s. Let F_i be the distribution of the i^{th} spare and t_i the replacement time with i spares remaining. A total of n spares are available and we assume $t_1 = \infty$. Define $k(t) \in \{1,\ldots,n\}$ so that:

$$\sum_{i=k(t)+1}^{n} t_i \leq t < \sum_{i=k(t)}^{n} t_i, \quad 0 \leq t < \infty.$$

Then $$P\{T_s > t\} = \left\{\prod_{i=k(t)+1}^{n} \bar{F}_i(t_i)\right\}\left\{\bar{F}_{k(t)}\left(t - \sum_{i=k(t)+1}^{n} t_i\right)\right\}, \quad 0 \leq t < \infty.$$

We use the convention that if the lower limit of a sum (product) is smaller than the upper limit then the sum (product) is 0 (1). The first term in the product is the probability of successful replacements and the second term is the probability of the newest spare surviving long enough to bring the total system lifetime up to t.

In the Complete Information Model, the lifetime distributions are identical so $F_i = F$ for $i = 1, \ldots, n$. For distributions with a known percentile, all of the minimizing distributions are also identical, as given in Chapter 3. In the case of distributions with known mean and variance, three cases must be considered. If $0 < t_i < \infty$, the minimizing distribution is the two-point distribution with $y_1 = t_i$

and $y_2 = \beta_{t_i}$. In the case, $t_i = 0$, F_i can be any distribution for which $F_i(0) = 0$. In the case, $t_i = \mu$, we set $t_i = \infty$ and any distribution in $C_{\mu,\sigma}$ works as a minimizing distribution for this case, so one can be chosen arbitrarily for computing $P\{T_s > t\}$.

Limiting Results

As the number of spares goes to infinity we will be primarily concerned with relating periodic schedules to general schedules. Before presenting the actual results, we will give the definition of the minimax periodic expected lifetime and schedule for multiple identical spares. The first step is to provide an analog of $L_v(t,F)$.

Definition 4.1: $L^n(t,F)$ = the expected lifetime with n spares using the periodic schedule t, under the lifetime distribution F. That is, $L^1(t,F) = L_0(t,F)$, $L^{n+1}(t,F) = L_{L^n(t,F)}(t,F)$, $n \geq 1$. □

Definition 4.2: Let $\gamma_D(n) = \sup_t \inf_{F \in D} L^n(t,F)$

and $\tau_D(n) = \operatorname{argsup}_t \inf_{F \in D} L^n(t,F)$. □

Thus, $\gamma_D(n)$ is the minimax periodic expected lifetime and $\tau_D(n)$ is the minimax periodic replacement time, both for n spares, for spares with distributions from the class, $D \subset C$.

It is quite easy to compute $L^n(t,F)$.

Proposition 4.1: For $n = 1, 2, 3, \ldots,$

$$L^n(t,F) = \left(\int_0^t \bar{F}(x)dx\right) \sum_{j=0}^{n-1} \bar{F}^j(t) .$$

Proof: This is evident from backsubstituting into the equations given in the definition of $L^n(t,F)$. □

Proposition 4.2:

$$L^\infty(t,F) \equiv \lim_{n\to\infty} L^n(t,F) = \begin{cases} \frac{1}{F(t)} \int_0^t \bar{F}(x)dx, & F(t) > 0, \\ \infty, & F(t) = 0. \end{cases}$$

Proof: This is evident from Proposition 4.1. □

Derman et al. prove two limiting results, given in Propositions 2.4 and 2.6. The analogous results for the Minimax Model are: $\lim_{n\to\infty} v_n = \bar{v}$, $\lim_{n\to\infty} v_n = \lim_{n\to\infty} v_D(n)$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \tau_D(n)$. The first result can be proven for all classes of distributions $D \subset C$.

Theorem 4.4: Given $h \in H$, define $v_0 = 0$, $v_{n+1} = h(v_n)$, for $n = 1, 2, 3, \ldots$. Then $\lim_{n\to\infty} v_n = \bar{v}$.

Proof: Let $v = \lim_{n\to\infty} v_n$, which exists since v_n is an increasing sequence. Suppose $v < \bar{v}$. Then $h(v) > v$. Let $\Delta = (h(v)-v)/2$. Then $h(v-\Delta) > v$, since $h(v) - h(v-\Delta) \leq \Delta$ by condition 3 of the definition of H. But there exists an n so that $v_n > v - \Delta$, which implies that $v_{n+1} > v$, which is a contradiction.

Now suppose that $v > \bar{v}$. Then there exists an n so that $v_n \leq \bar{v}$ and $h(v_n) > \bar{v}$. But this is impossible since for $v = (v_{n+1} + \bar{v})/2$, $h(v) = v$, but $v > v_n$ and $h(v_n) = v_{n+1} > v = h(v)$, which contradicts the monotonicity of h. $\Box$

The formulas for $\bar{v}_{p,s}$ and $\bar{v}_{\mu,\sigma}$ have been given in Chapter 3. When D is a single distribution function, $\bar{v} = \lim_{n\to\infty} v_n$ can be finite or infinite as stated in Proposition 2.4.

Our analog of Proposition 2.6 must be proved on a case by case basis. As usual, we will begin with the class of distributions determined by a known percentile. Let $D = \{F \in C \mid F(s^-) = p\}$. As might be expected, everything is easy in this case.

Proposition 4.4: Given D as defined above,

$$\gamma_D(n) = v_n \quad \text{and} \quad \tau_D(n) = t_n, \quad n = 1, 2, 3, \ldots ,$$

for v_n and t_n as given in Proposition 4.1.

Proof: The minimax schedule given in Proposition 4.1 is actually a periodic schedule, since t_n does not depend on n. $\Box$

From this, it is trivially true that $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \gamma_D(n)$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \tau_D(n)$.

We now return to the case of known mean and variance. In this case, the computation of $\gamma_D(n)$ and $\tau_D(n)$ is difficult. This is because Hoeffding's result, as used in Lemma 3.1, cannot be applied here, since there is no function $K_{t,v}(x)$ which satisfies condition 1 given in Lemma 3.1. Thus we cannot resort to three-point distributions

in order to solve this problem for a finite n. However, in the limit, we can use three-point distribution results.

Theorem 4.5: $\lim_{n\to\infty} \gamma_{\mu,\sigma}(n) = \bar{\bar{v}}_{\mu,\sigma}$.

This theorem not only gives the minimax periodic schedule lifetime, but states that it is the same as the limiting lifetime for nonperiodic schedules. So, once again, we obtain the result that, in the limit, it is no worse to use a periodic schedule. In order to prove this theorem, we will need two lemmas, which are now given. These will calculate the limiting values for periodic schedules under different assumptions. We will then show that the value of interest is bounded by these two values.

Lemma 4.2: $\lim_{n\to\infty} \sup_t \inf_{F\in C^3_{\mu,\sigma}} L^n(t,F) = \bar{\bar{v}}_{\mu,\sigma}$.

Proof: We will first see that the limit may be pulled past the supremum and infimum. Since $\inf_{F\in C^3_{\mu,\sigma}} L^n(t,F)$ is increasing in n for any $t \geq 0$, the limit and supremum may be exchanged. The interchange of the limit and infimum can be justified by noting that, for fixed t, $L^n(t,F)$ converges to $L^\infty(t,F)$ uniformly in F, provided we add the restriction that $F(t) > \varepsilon$ which can be done without loss of generality for small enough ε. Thus we are left with the problem:

$$\sup_t \inf_{F\in C^3_{\mu,\sigma}} L^\infty(t,F)$$

If we let F be the three-point distribution (y_1, y_2, y_3), then

$$L^{\infty}(t,F) = \begin{cases} \infty, & t < y_1 \\ (y_1p_1 + (1-p_1)t)/p_1, & y_1 \le t < y_2 \\ (y_1p_1 + y_2p_2 + p_3t)/(p_1+p_2), & y_2 \le t < y_3 \\ \mu, & y_3 \le t \end{cases}$$

As in Lemma 3.2, four cases must be analyzed.

Case 1: $t < y_1$

As in Lemma 3.2, the distribution, F, will always "choose" $y_1 \le t$ to avoid the high expected lifetime afforded by case 1.

Case 2: $y_1 \le t < y_2$

In this case, $L^{\infty}(t,F) = (y_1-t) + t/p_1$. Thus, $\partial L/\partial p_1 \le 0$ and $\partial p_1/\partial y_2 \ge 0$, so $\partial L/\partial y_2 \le 0$ and we must increase y_2 as much as possible. This leaves us the two-point distribution (y_1,y_2) with $y_1 \le t < y_2$ and $p_1 = \sigma^2/(\sigma^2+(\mu-y_1)^2)$. Thus $L^{\infty}(t,F) = y_1+t(\mu-y_1)^2/\sigma^2$ for this case.

Case 3: $y_2 \le t < y_3$

Now $L^{\infty}(t,F) = ((t-y_3)p_3 + \mu)/(1-p_3)$. In order to ensure $L^{\infty}(t,F) > \mu$, we need $\mu + t - y_3 \ge 0$ which is true if and only if

$$t \in \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - \sigma^2} \qquad (*')$$

As in Lemma 3.2, $\partial L/\partial y_3 < 0$ so we increase y_3 as much as possible, forcing $y_1 \to 0$ and leaving the two point distribution (y_2,y_3). Since $y_2 \le t < y_3$, we are in the same situation as case 2.

Case 4: $y_3 \leq t$

This case can never apply since we have already seen that $t \leq \mu$ is forced by (*') and $y_3 > \mu$.

We now see that $\inf_{y \leq t} \{y + t(\mu-y)^2/\sigma^2\} = t + t(\mu-t)^2/\sigma^2$. Notice that $d/dy = 1 - 2t(\mu-y)/\sigma^2$ is negative when (*') holds. Since we have the constraint, $y \leq t$, the infimum is attained at $y = t$. Thus, $\inf_{F\in C^3_{\mu,\sigma}} L^{\infty}(t,F) = t(\sigma^2 + (\mu-t)^2)/\sigma^2$ when (*') holds.

Now that all the cases have been considered, we need only one more step. Notice that $\sup_t t(\sigma^2 + (\mu-t)^2)/\sigma^2$ is the same problem we solved to find $\bar{v}_{\mu,\sigma}$ in Theorem 3.4. Thus,

$$\sup_t \inf_{F\in C^3_{\mu,\sigma}} L^{\infty}(t,F) = \bar{v}_{\mu,\sigma}. \square$$

Before proving the next lemma we need to make one definition which will be used in the lemmas to follow. Let $u_0(t) \equiv 0$, $u_{n+1}(t) = \inf_{F\in C_{\mu,\sigma}} L_{u_n(t)}(t,F)$, and $u_n = \sup_t u_n(t)$. Thus u_n is the minimax expected lifetime for a slightly different interpretation of a periodic schedule. It is not required that the spares have the same distribution, only that all of the distributions have the same mean and variance.

<u>Lemma 4.3</u>: $\lim_{n\to\infty} u_n = \bar{v}_{\mu,\sigma}$.

<u>Proof</u>: Since $u_{n+1}(t) \geq u_n(t)$ for $n \geq 1$, $t \geq 0$, it follows that $\lim_n \sup_t u_n(t) = \sup_t \lim_n u_n(t)$. Thus, $\lim_{n\to\infty} u_n = \sup_t \lim_{n\to\infty} u_n(t)$. If we choose a t such that $\inf_{F\in C_{\mu,\sigma}} L_{\mu}(t,F) > \mu$ then $u_{n+1}(t) > u_n(t)$ for all n, so, for such a t,

$$u_{n+1}(t) = \inf_{F\epsilon C_{\mu,\sigma}} L_{u_n(t)}(t,F) = t + \frac{u_n(t)(\mu - t)^2}{\sigma^2 + (\mu-t)^2}$$

By induction, $u_n(t) = t(1 + a + a^2 + \ldots + a^{n-1})$, where $a = (\mu-t)^2/(\sigma^2+(\mu-t)^2)$. Thus, $\lim_{n\to\infty} u_n(t) = t/(1-a) = t(\sigma^2+(\mu-t)^2)/\sigma^2$. But now, $\sup_t (\lim_{n\to\infty} u_n(t))$ is the same problem as in Lemma 4.2 and Lemma 3.4, so this lemma is proved. □

We now have the necessary results to prove Theorem 4.5.

Theorem 4.5: $\lim_{n\to\infty} \gamma_{\mu,\sigma}(n) = \bar{v}_{\mu,\sigma}$.

Proof: We will see that:

$$\bar{v}_{\mu,\sigma} = \lim_n \sup_t \inf_{F\epsilon C^3_{\mu,\sigma}} L^n(t,F) \geq \lim_n \gamma_{\mu,\sigma}(n) \geq \lim_n u_n = \bar{v}_{\mu,\sigma}.$$

The first and last equalities are proved in Lemmas 4.2 and 4.3. The inner inequalities are actually true for every n, not just in the limit. The left-hand inequality is easy to see. Since $C^3_{\mu,\sigma} \subset C_{\mu,\sigma}$,

$$\inf_{F\epsilon C^3_{\mu,\sigma}} L^n(t,F) \geq \inf_{F\epsilon C_{\mu,\sigma}} L^n(t,F).$$

Now we take supremums on both sides and the right-hand side becomes $\gamma_{\mu,\sigma}(n)$.

The fact that $\gamma_{\mu,\sigma}(n) \geq u_n$ follows from the fact that each of the n spares represented by u_n may have a different distribution, while all n spares must have the same distribution in $\gamma_{\mu,\sigma}(n)$. Thus, the minimization of u_n is done over a larger space. □

Since the same limiting optimal replacement time results in Lemmas 4.2 and 4.3, it is strongly suggested that this should be the limiting value of $\tau_{\mu,\sigma}(n)$ as $n \to \infty$. But no proof of this proposition is given. Further, this limiting replacement time in Lemmas 4.2 and 4.3 is equal to $\lim_{n\to\infty} t_n$.

Computational Example

Now that we have successfully treated the finite and infinite case of multiple spares, we will give a numerical example comparing all models discussed so far. We will give v_n and t_n for selected values of n for three different spares.

Example 4.1: The first of the three spares is the one used in the example in Chapter 2. It has a known distribution with density, $f(t) = 2t$, $0 \leq t \leq 1$. The second spare has known mean and variance which correspond to the mean and variance of the first spare. The second spare has an expected lifetime of 2/3 with variance 1/18. The final spare has a known percentile, corresponding to the first quartile of the distribution of the first spare.

The information assumed about each of the spares is presented in Table 4.1 and the schedules and lifetimes are given in Table 4.2. For the minimax spares, the predicted lifetime (based on the information assumed) is given in the column labeled v_n. The expected lifetime which would be realized if the actual lifetime distribution is that of spare 1 is given in the column labeled r_n. □

Spare #	Spare Type	Information
1	complete information	$f(t) = 2t,\ 0 \leq t \leq 1$
2	minimax, known mean and variance	$\mu = 2/3,\ \sigma^2 = 1/18$
3	minimax, known percentile	$s = 1/2,\ p = 1/4$

Table 4.1 Spare Definitions

	Spare 1		Spare 2			Spare 3		
n	v_n	t_n	v_n	r_n	t_n	v_n	r_n	t_n
1	0.67	1.00	0.67	0.67	∞	0.38	0.67	∞
2	0.96	0.54	0.78	0.92	0.36	0.66	0.96	0.50
3	1.19	0.43	0.85	1.14	0.33	0.87	1.18	0.50
4	1.38	0.37	0.90	1.33	0.31	1.03	1.34	0.50
5	1.55	0.33	0.94	1.50	0.30	1.14	1.46	0.50
10	2.20	0.23	1.01	2.17	0.27	1.42	1.75	0.50
20	3.14	0.16	1.03	2.94	0.27	1.50	1.83	0.50
40	4.45	0.11	1.03	3.48	0.27	1.50	1.83	0.50

Table 4.2 Optimal Schedules and Expected Lifetimes

Notice that the first spare has the longest expected lifetime for every n. This is because it has the best information available on the actual distribution of the spare lifetimes. The schedule generated by spare 2 gives the next largest realized expected lifetime for $n \geq 5$ even though it predicts a smaller expected lifetime than the third spare for $n \geq 3$. Also note that the realized expected lifetimes for the minimax spares are quite a bit larger than the predicted expected lifetimes. This is because the minimax approach is conservative.

Chapter 5

Different Spares

In the previous chapter, we treated the case of multiple, identical spares. Since the spares were identical, we did not have to concern ourselves with the problem of deciding the order in which the spares should be used. In this chapter, we will allow the spares to be different from each other and consider the sequencing problem.

Since we are now capable of computing $h_D(v)$ for a variety of classes, D, we could always check all of the $n!$ possible orderings; computing the expected lifetime of each and simply choosing the sequence which provides the largest expected lifetime. However, since $n!$ grows so quickly, this would limit us to solving problems with only a small number of spares. Two improvements on this total enumeration scheme are the topic of this chapter.

The first improvement is based upon a sufficient condition for an optimal sequence to exist in which a particular spare precedes another spare. Before discussing this in more detail, a formal definition of the optimal sequencing problem is given.

Definitions

Definition 5.1: Given a set of functions, $h_1, h_2, \ldots, h_n \in H$ and a permutation, π, of $\{1,2,\ldots,n\}$, let

$$P(\pi,v) = h_{\pi_1}(h_{\pi_2}(\ldots h_{\pi_n}(v) \ldots)).$$

The optimal sequencing problem is

$$\max_\pi P(\pi,v). \quad \square$$

Notice that we have defined the optimal sequencing problem without any reference to actual spares. This is because the spares are now represented by their single spare return functions. For example, if the third spare in our list had a known lifetime distribution function, F, then we would have $h_3(v) = h_F(v)$. Further, $P(\pi,v)$ is the expected lifetime with the permutation π and reward v.

The sequencing problem is defined for functions which belong to H, but this is done only for simplicity. This problem could just as easily be defined for arbitrary functions. However, all of our single spare return functions do belong to H, and later in this chapter, it will be important to restrict ourselves to the class H.

As noted previously, only $n!$ distinct permutations, π, exist, so the optimal sequencing problem may be solved by computing $P(\pi,v)$ for every π and choosing the permutation yielding the largest value. While our algorithm will be based on this idea, two ways of reducing the number of permutations to be checked is discussed.

Before describing an improved algorithm, some additional definitions are provided.

Definition 5.2: If $\pi^{-1}(i) < \pi^{-1}(j)$ then we say "i precedes j in π," or "h_i precedes h_j in π." □

For example, for $n = 3$, if $\pi_1 = 2$, $\pi_2 = 3$, and $\pi_3 = 1$, then $P(\pi,v) = h_2(h_3(h_1(v)))$ and 2 precedes 3 which precedes 1. This corresponds to using spare 2 first and spare 1 last, after which we receive v if all replacements have been made successfully.

Definition 5.3: Given $g,h \in H$, if

1. $g(v) \geq h(v)$ for all $v \geq 0$, and
2. $g(v) - g(u) \geq h(v) - h(u)$ for all $v > u \geq 0$ such that

$h(v) > v$, then we say "g is better than h" or $g \gg h$. □

Note that if $g = h$, then $g \gg h$ and $h \gg g$. A function g is "strictly better" than h if $g \gg h$ and $g \neq h$. Also note that $\gg$ is transitive, so the relation, $\gg$, defines a partial order on a set $G \subseteq H$.

The relation, "better than" will form the cornerstone of the improved algorithm to be presented. For now, we will pass over the question of how to determine if a particular single spare return function is better than another in favor of presenting the main theorem of this chapter.

Sequencing Theorem

Theorem 5.1: Suppose we are given $h_1, \ldots, h_n \in H$ and $v \geq 0$. If $h_1 \gg h_2$, then there exists an optimal permutation in which h_1 precedes h_2.

This can be stated roughly as "always use the best spares first." Thus, if we are given a list of pairs of functions which are ordered by $\gg$, then many permutations can be ruled out, and the expected lifetime calculations may be carried out for a smaller number of permutations.

Before giving the actual proof of this theorem, we will prove two lemmas. The first lemma proves that H is closed under composition. The second lemma is essentially a proof of Theorem 5.1 for $n = 3$. Finally, these two lemmas will be combined into the complete proof of the theorem.

<u>Lemma 5.1</u>: Given $g, h \in H$, $h \circ g \in H$.

<u>Proof</u>: This will be done in three steps, one for each condition which $h \circ g$ must satisfy.

1. $g(h(v)) \geq h(v) \geq v$.

2. $v \geq u$ implies that $h(v) \geq h(u)$ implies that $g(h(v)) \geq g(h(u))$.

3. $v \geq u$ implies that $h(u) - u \geq h(v) - v$, while $h(u) \leq h(v)$ implies that $g(h(u)) - h(u) \geq g(h(v) - h(v)$. Adding these two inequalities proves that $g(h(u)) - u \geq g(h(v)) - v$ as required. □

<u>Lemma 5.2</u>: Given $f, g, h \in H$ with $f \gg g$ and $v \geq 0$, then

either $g(h(f(v))) \leq f(h(g(v)))$,

or $g(h(f(v))) \leq h(f(g(v)))$.

That is, there exists a sequence in which f precedes g which gives a better expected lifetime than the sequence with g preceding h preceding f.

Proof: Let $\Delta = f(v) - g(v) \geq 0$ and $u = g(v)$ so $f(v) = u + \Delta$. Since $h \in H$, $h(u+\Delta) \leq h(u) + \Delta$. Thus, $g(h(u+\Delta)) \leq g(h(u) + \Delta)) \leq g(h(u)) + \Delta$. If $g(h(u)) > h(u)$ then, since $f \gg g$ and $h(u) \geq v$, $f(h(u)) - g(h(u)) \geq f(v) - g(v) = \Delta$. From this it follows that $g(h(f(v))) = g(h(u+\Delta)) \leq f(h(u)) = f(h(g(v)))$. On the other hand, if $g(h(u)) = h(u)$, then $g(h(f(v))) = h(f(v)) \leq h(f(g(v)))$, which completes the proof. □

Corollary 5.1: $f,g \in H$, $f \gg g$ implies that

$$f(g(v)) \geq g(f(v)) \quad \text{for all} \quad v \geq 0.$$

Proof: Take h in the previous theorem to be the identity function, $h(v) = v$. □

We can now give a complete proof of Theorem 5.1.

Theorem 5.1: Suppose we are given $h_1, \ldots, h_n \in H$ and $v \geq 0$. If $h_1 \gg h_2$, then there exists an optimal permutation in which h_1 precedes h_2.

Proof: We will see that, given any permutation π in which h_2 precedes h_1, we can find a permutation, π', which is at least as good as π, in which h_1 precedes h_2.

First divide the permutation π into five (possibly empty) subpermutations. These five subpermutations are: the functions which precede h_2, h_2, the functions which precede h_1 but are preceded by h_2, h_1, the functions which are preceded by h_1. By Lemma 5.1, each of these five groups is equivalent to a single function in H. If a group

is empty, then represent it by the identity function $g(x) = x$, which is in H. Thus, we now have

$$P(\pi,v) = g_1(g_2(g_3(g_4(g_5(v))))),$$
$$\text{where } g_2 = h_2 \text{ and } g_4 = h_1.$$

Since these functions are nondecreasing, if $f(v) \geq g_2(g_3(g_4(v)))$ then $g_1(f(g_5(v))) \geq P(\pi,v)$. But by Lemma 5.2, since $g_4 \gg g_2$, either

$$g_3(g_4(g_2(v))) \geq g_2(g_3(g_4(v))), \text{ or}$$
$$g_4(g_3(g_2(v))) \geq g_2(g_3(g_4(v))).$$

Thus, for any $v \geq 0$, there is a permutation, π' with h_1 preceding h_2 so that $P(\pi',v) \geq P(\pi,v)$. □

<u>Corollary 5.2</u>: If $h_1 \gg h_2 \gg \ldots \gg h_n$, then $\pi_i = i$ is an optimal sequence for any v.

<u>Proof</u>: By Theorem 5.1, no other sequence could be strictly better. □

Thus, if the set of functions is completely ordered, then it is trivial to find an optimal sequence.

The following definition gives a name to the permutations which do not violate the result of Theorem 5.1.

<u>Definition 5.4</u>: A permutation will be called feasible if no function precedes a strictly better function. □

In light of this definition, a shorter statement for Theorem 5.1 would be that the best feasible permutation is an optimal permutation. The next proposition provides a bound on the number of feasible permutations.

Proposition 5.1: Suppose $\{h_1, h_2, \ldots, h_n\}$ can be broken down into k disjoint sets, $I_1, \ldots, I_k$, such that each set is completely ordered and set I_i contains n_i elements, $i = 1,2,\ldots,k$. Then at most

$$\frac{n!}{n_1!\ n_2!\ \ldots\ n_k!}$$

permutations are feasible.

Proof: Functions from within a set can appear in a feasible permutation in only one order, but, in general, functions from different sets may appear in any order. The actual result is a standard combinatorial fact. □

Consider the following numerical example. If $n = 10$ and we checked all permutations, then $10! = 3{,}628{,}800$ evaluations would have to be done. However, if two completely ordered groups of size 5 exist, then only $10!/5!5! = 252$ permutations need to be checked.

At this point, it is tempting to suggest that, for $f,g \in H$, if $f(g(v')) > g(f(v'))$ for some $v' > 0$, then $f(g(v)) > g(f(v))$ for all $v > 0$. That is, if one spare is better than another for some v, then maybe it actually is better for all v. Following is a counterexample to this claim. We will consider two spares with known mean and

variance. Suppose the first spare has an expected lifetime of 1 with a standard deviation of .2, and the second spare has an expected lifetime of 2 with a standard deviation of .4. Let $\pi_1 = \{1,2\}$ and $\pi_2 = \{2,1\}$. Then, for $v = 0$, $P(\pi_1,v) = 2.22$ while $P(\pi_2,v) = 2.11$. But, for $v = 3.7$, $P(\pi_1,v) = 4.21$, while $P(\pi_2,v) = 4.23$. Thus depending on the value of v, different orders may be optimal.

Better Spares

Before we can apply Theorem 5.1, we need to know when a particular single spare return function is better than another. As an example, we will consider known percentile minimax spares. Then we will move on to the more interesting cases of spares with completely known distributions and minimax spares with known mean and variance.

Proposition 5.2: h_{p_1,s_1} is better than h_{p_2,s_2} when $p_1 \le p_2$ and $s_1 \ge s_2$.

Proof: For the duration of this proof, let $h_i = h_{p_i,s_i}$, for $i = 1,2$. We need to verify two conditions.

1. $h_1(v) \ge h_2(v)$ since $(1-p_1) \ge (1-p_2)$, $s_1 \ge s_2$, and $s_1(1-p_1)/p_1 \ge s_2(1-p_2)/p_2$.

2. $h_2(v) > v$ implies that $0 \le v < s_2(1-p_2)/p_2$, in which case $v < \bar{v}_1$ and $h_1'(v) = 1-p_1 \ge 1-p_2 = h_2'(v)$. □

As usual, the percentile case was handled easily. The next two theorems provide sufficient conditions for one spare to be better

than another for spares with known lifetime distributions and for spares with known mean variance.

Theorem 5.3: Let F and G have continuous densities, and assume $\bar{F}(t) \geq \bar{G}(t)$, $t \geq 0$. If $\bar{F}(t_F(v)) \geq \bar{G}(t_G(v))$ for all v, then $h_F \gg h_G$.

Proof: First, $\bar{F} \geq \bar{G}$ ensures that $L_v(t,F) \geq L_v(t,G)$ for all t and v, so $h_F(v) \geq h_G(v)$ for all $v \geq 0$. Now,

$$h_F'(v) = t_F'(v)(\bar{F}(t_F(v)) - vf(t_F(v))) + \bar{F}(t_F(v)).$$

But, since $t_F(v)$ is optimal, $\bar{F}(t_F(v)) = vf(t_F(v))$, so $h_F'(v) = \bar{F}(t_F(v))$ and $h_F'(v) \geq h_G'(v)$ if and only if $\bar{F}(t_F(v)) \geq \bar{G}(t_G(v))$. $\square$

Theorem 5.4: If $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$, then $h_{\mu_1,\sigma_1} \gg h_{\mu_2,\sigma_2}$.

Proof: Let $h_i = h_{\mu_i,\sigma_i}$ for $i = 1, 2$. In Theorem 4.1, it was proven that $h_{\mu,\sigma}$ is increasing in μ and decreasing in σ, provided $h_{\mu,\sigma}(v) > v$. If $\bar{\mu}_2 < \bar{v}_2$ then $\bar{\mu}_1 = \min\{2\sigma_1,\mu\} \leq 2\sigma_2 = \bar{\mu}_2$ and $\bar{v}_1 \geq \bar{v}_2$, so that, if $h_1(v) = \max\{\mu_1,v\}$ then $h_2(v) = \max\{\mu_2,v\} \leq \max\{\mu_1,v\}$. Thus, $h_1 \geq h_2$. If $\bar{\mu}_2 = \bar{v}_2$ then $h_1 \geq h_2$ since $\mu_1 \geq \mu_2$.

To see that $h_1'(v) \geq h_2'(v)$ when $h_2(v) > v$ ($v \leq \bar{v}_2$), recall from the proof of Lemma 4.1: $h_{\mu,\sigma}'(v) = (\mu-t_{\mu,\sigma}(v))^2/(\sigma^2+(\mu-t_{\mu,\sigma}(v))^2)$ when $h_{\mu,\sigma}(v) > v$. So $h_{\mu,\sigma}'$ is constant in μ, since $\mu - t_{\mu,\sigma}$ is constant in μ, and decreasing in σ as required. Once again we have used the fact that $\bar{v}_1 \geq \bar{v}_2$. $\square$

Thus, for spares with known mean and variance, "better than" can be interpreted in a very natural sense.

Eliminating Spares

One final technique can be used which may further reduce the number of permutations which need to be checked. Suppose we were given a spare along with its distribution. If we then discovered that this distribution was NWUE, then, by Proposition 2.1, we would know that using this spare before other spares could not strictly help us except through its mean. More precisely:

Theorem 5.5: Given $h_1, \ldots, h_n \in H$ and $h_1(v) = \max\{v, h_1(0)\}$ for all $v \geq 0$. Then, for any $v \geq 0$, there exists an optimal sequence in which h_1 precedes no other spares.

Proof: Suppose h_1 directly precedes h_2. Then $h_1(h_2(v)) = \max\{h_2(v), h_1(0)\} \leq h_2(h_1(v))$. Thus h_1 need not precede any other spare. □

Thus, a spare with $\bar{\mu} = \bar{v}$ can contribute only through its mean. If two or more spares have this property, then only the one with the largest mean need be used. Further, it may be possible to identify other spares which need not be used.

Theorem 5.6: If $\bar{\mu}_1 = \bar{v}_1$ and $\bar{v}_2 \leq \mu_1$ then no advantage may be gained through the use of the second spare.

Proof: From Theorem 5.5, we know that we would get no advantage in having spare 1 precede spare 2. However, if spare 2 precedes spare 1, then we also get no advantage since the reward for successful replacement of spare 2 would be at least $\mu_1 \geq \bar{v}_2$. □

Computational Example

Example 5.1: Suppose we have five spares which need to be scheduled. We know the distribution of the first spare, and it has the density, $f(t) = 2t$, $0 \leq t \leq 1$. For the second, third, and fourth spares, we know only their mean and variance. Specifically, the second spare has mean 1 and standard deviation .4. The third spare has mean 1.1 and standard deviation .35. The fourth spare has mean 2/3 and standard deviation 2/3. The fifth and last spare has its median at time 1/2. This information is summarized in Table 5.1. Since no other spares are available to us, we will take $v = 0$.

Spare #	Spare Type	Information
1	complete information	$f(t) = 2t$, $0 \leq t \leq 1$
2	minimax, known mean and variance	$\mu = 1$, $\sigma = .4$
3	minimax, known mean and variance	$\mu = 1.1$, $\sigma = .35$
4	minimax, known mean and variance	$\mu = 2/3$, $\sigma = 2/3$
5	minimax, known percentile	$s = 1/2$, $p = 1/2$

Table 5.1 Spare Definitions

In deciding on the optimal sequence for these spares, three things should be noted. The first is that the fourth spare has $\bar{\mu} = \bar{v}$ since the mean is less than twice the standard deviation. This means that this spare must be scheduled last. The next important fact is that the fifth spare has $\bar{v} = 1/2$, which is less than 2/3, the mean of the fourth spare. Thus, using the fourth spare cannot help us at all. The final important fact to note is that the third spare is better than the second spare, so the third spare precedes the second spare in any feasible sequence.

Permutation	$P(\pi,0)$
$\pi_1 = 1,\ \pi_2 = 3,\ \pi_3 = 2,\ \pi_4 = 4$	1.45
$\pi_1 = 3,\ \pi_2 = 1,\ \pi_3 = 2,\ \pi_4 = 4$	1.42
$\pi_1 = 3,\ \pi_2 = 2,\ \pi_3 = 1,\ \pi_4 = 4$	1.33

Table 5.2 Permutation Evaluations

Thus, only three sequences remain to be checked. They are listed in Table 5.2 along with their expected lifetimes. From examining these expected lifetimes, it is apparent that the first sequence listed is an optimal sequence. □

Chapter 6

Extensions

This chapter will treat two extensions to the Basic Model, which has been the focus of our work so far. These extensions are not simply different assumptions about the lifetimes of the spares but are structural variations of the Basic Model. In both, it is assumed that the complete lifetime distribution of the identical spares is known.

Spares in Parallel

The first extension to be discussed deals with the idea of using spares in parallel. As such it will be called the Parallel Model.

Parallel Model: The spares are aggregated into arbitrarily-sized groups and these groups are scheduled in an arbitrary sequence. The failure time of a group is the largest failure time of the spares in the group. As in the Basic Model, the lifetime of the entire system is the time until the first failure of an installed group.

The first thing to notice is that the Parallel Model yields expected lifetimes which are no worse than the Complete Information Model because we could choose each of our groups to have only one spare. It is tempting to think that it would always be best to have one

large group of spares, since this would minimize the chance of a very early failure. However, when the spares are reliable early in their lifetimes, this would waste quite a bit of potential expected lifetime by overlapping the reliable portions of the spares' lifetimes.

We now consider the problem of determining the single spare return function for a group of n spares. As in Chapter 2, let F be the cumulative distribution function of the lifetime of a single spare. Also, let μ be the expected lifetime of a spare. Let $F^{(n)}$ denote the cumulative distribution function of the lifetime of a group of n spares. Since the group fails if and only if every component in the group fails, $F^{(n)} = F^n$. So, if h_n is the single spare return function for a group of n spares,

$$h_n(v) = \sup_t L_v(t,F^n).$$

It is a simple calculation to find the failure rate for a group of n spares:

$$r_n(t) = \frac{nF^{n-1}(t)}{1 - F^n(t)} .$$

As stated in Chapter 2, the optimal replacement time, $t_n(v)$, satisfies $r_n(t_n(v)) = 1/v$. The actual solution of $r_n(t) = 1/v$ to find $t_n(v)$ and $h_n(v)$ must be undertaken on a case by case basis depending on the spare lifetime distribution, F. If F is IFR then $F^{(n)}$ is also IFR. See Barlow and Proshan [1965] pp. 38-39 for a proof of this fact. In this case, the equation, $r_n(t) = 1/v$, is easy to solve numerically using a one-dimensional search.

Assuming that we can solve for $h_n(v)$, the optimal grouping and sequencing for a set of spares may be found by the following method. Let u_n be the optimal expected lifetime with n spares. Assume u_k is known for $1 \leq k \leq n$, and $u_1 = \mu$. Then u_{n+1} may be calculated according to the following equation.

$$u_{n+1} = \sup_{1 \leq i \leq n} h_i(u_{n+1-i}), \quad n \geq 1.$$

In this expression, i represents the size of the initial group, and $h_i(u_{n+1-i})$ is the expected lifetime using a group of size i followed by the optimal use of the remaining $n+1-i$ spares.

Thus, if u_k is known for $1 \leq k \leq n$, then h_i must be evaluated once for each $i \in \{1, \ldots, n\}$ in order to find u_{n+1}.

The next theorem says that no matter what lifetime distribution we have, provided it is not degenerate at 0, the limiting expected lifetime is infinite.

Theorem 6.1: $\lim_{n\to\infty} u_n = \infty$, provided $F(0) < 1$.

Proof: Since $F(0) < 1$, there exists $\varepsilon > 0$ so that $F(\varepsilon) \equiv p < 1$. Let $w_n(m)$ be the expected lifetime using the periodic schedule, ε, on n groups of size m each. Then $\lim_{n\to\infty} w_n(m) \geq \varepsilon(1-p^m)/p^m$ since $(1-p^m)/p^m$ is the expected number of groups which survive before the first failure. Clearly, $\lim_{n\to\infty} w_n(m)$ can be made as large as necessary by choosing m large enough. □

We will now consider an example of the calculation of the optimal grouping and scheduling of spares.

Example 6.1: Let the distribution of the spares be uniform on $[0,1]$, so $F(t) = t$, $0 \leq t \leq 1$. Then $F^{(n)}(t) = F^n(t) = t^n$, $t_n(v)$ satisfies:

$$t_n(v) + vnt_n(v)^{n-1} - 1 = 0,$$

and $$h_n(v) = t_n(v) - \frac{t_n(v)^{n+1}}{n+1} + (1 - t_n(v)^n)v.$$

Table 6.1 lists the optimal grouping and sequencing for n up to 8. The "sequence" column is read as follows: for $n = 3$, the optimal sequence is a group of size two followed by a group of size one. □

n	sequence	expected lifetime(u)
1	1	0.50
2	2	0.67
3	2→1	0.83
4	2→2	0.95
5	2→2→1	1.07
6	3→2→1	1.20
7	3→2→2	1.30
8	3→2→2→1	1.41

Table 6.1 Optimal Groupings and Expected Lifetimes

Note that in the example given, larger groups always precede smaller groups. It is not known if this is true in general. Since $\bar{F}^{(n)} \leq \bar{F}^{(n+1)}$, it may be that Theorem 5.3 could be applied to prove that h_{n+1} is better than h_n and that larger groups would

precede smaller groups. The problem lies in proving $\bar{F}^{(n+1)}(t_{n+1}(v)) \geq \bar{F}^{(n)}(t_n(v))$ for all v.

Also note that it is not true that $u_{n+1} - u_n \leq u_n - u_{n-1}$ in general, since $u_6 - u_5 > u_5 - u_4$. The reason for this seems to be the integrality of the spares. Thus, if the groups could consist of a non-integral number of spares, then perhaps this result would be true.

One More Spare

The second extension to be treated involves the idea of receiving an additional spare at some time in the future. This might be the case if the limited availability of spares is caused by long production times. For instance, we might have an additional component under construction but unavailable for a relatively long time. This model will be known as the Additional Spare Model.

Additional Spare Model: As in the Complete Information Model, n spares are available for immediate use, and one additional spare will become available t units of time from now. The extra spare cannot be used if the system has already failed before delivery of the extra spare. All other assumptions of the Complete Information Model apply.

Let $v(n,t)$ be the optimal expected lifetime with n spares available now and an additional spare available t units of time from now. If $t = 0$, then the additional spare has already been received. Thus $v(n,0)$ corresponds to v_n as defined in Chapter 2.

The following equations define $v(n,t)$ for all n and t.

$$v(0,t) = 0, \quad t \geq 0,$$

$$v(n,0) = \sup_y L_{v(n-1,0)}(y,F), \quad n = 1, 2, \ldots, \text{ and}$$

$$v(n,t) = \max\{\sup_{y<t} L_{v(n-1,t-y)}(y,F),$$

$$\sup_{y\geq t} L_{v(n,0)}(y,F)\}, \quad n = 1, 2, \ldots; \ t > 0.$$

The calculation of $v(n,0)$ is exactly the same as the calculation of v_n described in Chapter 2. The calculation of $v(n,t)$ for $t > 0$ is complicated by the fact that $v(n-1,s)$ must be known for all s in $[0,t]$. Thus, numerical solution is impractical and the calculations must be done analytically. Several monotonicity results are available.

Theorem 6.2: $v(n+1,0) \geq v(n,t), \quad n \geq 1, \ t > 0.$

Proof: The only difficult part of the proof is to show that $v(n,0) \geq v(n-1,t-y)$ for $0 \leq y \leq t$. This follows inductively. For $n = 0$, $v(1,0) = \mu \geq 0 = v(0,t)$. From $v(n,0) \geq v(n-1,t-y)$, $t-y > 0$, it follows that $v(n+1,0) \geq v(n,t)$, $t > 0$ by using the same schedule on $v(n+1,0)$ as is optimal for $v(n,t)$. □

Thus, it is better to have the extra spare right now than to wait for the spare. As the next theorem shows, it is also better to receive the spare sooner than later.

Theorem 6.3: $v(n,t_1) \geq v(n,t_2)$, $n \geq 1$, $t_1 \leq t_2$.

Proof: We will refer to $t = t_i$ as case i. If in case 1, we follow the optimal strategy from case 2, then we will eventually be in the situation that case 1 has received its extra spare while case 2 has not. This returns us to the situation of the previous theorem which completes this proof. □

It is also trivially true that additional spares are better, i.e. $v(n+1,t) \geq v(n,t)$, $t \geq 0$, which follows by choosing $y = 0$ in the optimization of $v(n+1,t)$.

The final result of this section is that the replacement time increases as the time until the additional spare is received increases. Thus, we extend our replacement times in the hope of actually receiving the additional spare before using up our current allotment of spares. Let $x(n,t)$ be the smallest optimal replacement time yielding $v(n,t)$, $n \geq 1$, $t \geq 0$.

Theorem 6.4: $x(n,t_1) \leq x(n,t_2)$, $n \geq 1$, $0 < t_1 \leq t_2$.

Proof: Let $x_i = x(n,t_i)$, $i = 1, 2$. Also, for $i = 1, 2$, let

$$v_i(y) = \begin{cases} v(n-1,t_i-y), & y < t_i, \\ v(n,0), & y \geq t_i. \end{cases}$$

Recall that $v_1(y) \geq v_2(y)$ and v_i is nondecreasing in y. Thus, $v_i(y)$ is the reward we receive for successful substitution after time y in case i. Now,

$$v(n,t_i) = L_{v_i(x_i)}(x_i,F), \quad i = 1, 2.$$

$$\text{So}\quad v(n,t_1) - v(n,t_2) \geq L_{v_1(x_1)}(x_2,F) - L_{v_2(x_2)}(x_2,F)$$
$$= \bar{F}(x_2)(v_1(x_1) - v_2(x_2)),$$

$$\text{and}\quad v(n,t_1) - v(n,t_2) \leq L_{v_1(x_1)}(x_1,F) - L_{v_2(x_2)}(x_1,F)$$
$$= \bar{F}(x_1)(v_1(x_1) - v_2(x_2)).$$

Thus, $\bar{F}(x_1)(v_1(x_1) - v_2(x_2)) \geq \bar{F}(x_2)(v_1(x_1) - v_2(x_2))$.

We will consider three cases on the sign of $v_1(x_1) - v_2(x_2)$.

Case 1: If $v_1(x_1) = v_2(x_2)$ then, either $x_1 \leq x_2$ or $x_2 < x_1$ and $v_1(x) = v_2(x) = c$ for $x \in [x_2,x_1]$. In this case, $x_1 > x_2$ contradicts their optimality since $L_{v_1(x)}(x,F)$ and $L_{v_2(x)}(x,F)$ agree for $x \in [x_2,x_1]$.

Case 2: If $v_1(x_1) < v_2(x_2)$ then $x_1 \leq x_2$ since $v_1 \geq v_2$ and the v_i are nondecreasing.

Case 3: Finally, if $v_1(x_1) > v_2(x_2)$, then $\bar{F}(x_1) \geq \bar{F}(x_2)$. Now, if $\bar{F}(x_1) > \bar{F}(x_2)$ then $x_1 \leq x_2$ since $\bar{F}$ is nonincreasing. If $\bar{F}(x_1) = \bar{F}(x_2)$ then it is possible that $x_2 < x_1$, but $x_2 < x_1$ would contradict their optimality since x_2 could be increased to x_1, strictly improving $L_{v_2(x)}(x,F)$. $\Box$

<u>Theorem 6.5</u>: $x(n+1,0) \leq x(n,t)$, $n \geq 1$, $t > 0$.

<u>Proof</u>: This proof is the same as in the previous theorem with the appropriate redefinition of v_1 and v_2. In effect, we now have $t_1 = 0$. $\Box$

It should be noted that if $x(n+1,0) > t$ then $v(n+1,0) = v(n,t)$ and $x(n+1,0) = x(n,t)$. This is simply the statement that, if we did not need to wait for the extra spare then we get the same schedule and expected lifetime; whether or not we have the spare right now.

In fact, this result can be extended to the following. If $t < x(n+1,0) + x(n,0) + \ldots + x(2,0)$ then $v(n+1,0) = v(n,t)$ and $x(n+1,0) = x(n,t)$. That is, we put together our normal schedule, and if the total time until we use the last spare is more than the time we must wait for our extra spare, then we can use the normal schedule and get the same return after waiting for the spare.

The importance of this fact is that it can make determining an optimal schedule much easier if the time until the extra spare is received is not very large compared to the replacement times when few spares are available.

Chapter 7

Conclusions

In the previous chapters, we have defined and discussed a model which is easier to implement than the spare replacement model presented by Derman et al. [1984]. The implementation of the minimax model with known mean and variance is easier for two reasons. The first is that less information need be known about each spare. In particular, we assume that only the mean and variance of the spare lifetime is known, in comparison to knowledge of the entire spare lifetime distribution required by Derman et al. Secondly, in our model it is simple to find optimal schedules and expected lifetimes through a sequence of one-dimensional searches described in Chapter 3. In Derman et al., this calculation depends on the particular lifetime distribution, and is easy for some distributions and very difficult for others.

Not only have we relaxed the requirement that the spare lifetime distribution be known, but the problem of sequencing different spares was considered in a general context in Chapter 5. Finally, two extensions to the Basic Model, the Extra Spare Model and the Parallel Model, have been presented and discussed. In all cases, computational methods have been provided for determining optimal schedules and expected system lifetimes.

Further Research

Further research may prove fruitful in a number of the following areas. The areas are divided into three categories: refinements, extensions, and new models, which will be discussed in this order.

One obvious possibility for refinement would be an algorithm for sequencing spares (Chapter 5) which does not rely on enumeration. The problem with the given algorithm is that, in the worst case, $n!$ permutations must be evaluated. It seems that an algorithm could exist which would have a better worst case performance than the given algorithm. Of course, the actual performance will depend on the actual spares which must be sequenced.

A related issue is that there may exist a better sufficient condition for one spare to precede another than $f \gg g$. If such a condition is found, then it could be the case that fewer permutations would have to be evaluated, although, in the worst case, $n!$ permutations would still have to be evaluated. According to the counterexample in Chapter 5, there does not exist a condition so that given any two functions, one of them always precedes the other. However, it is certainly possible that a condition is available which is more discriminating than $f \gg g$.

A reasonable extension to the assumption of known mean and variance in Chapters 3 and 4 is the knowledge of additional moments of the spares lifetime distribution. If, for example, we assumed knowledge of the

first k moments of the lifetime distribution, then Hoeffding's result could be applied to reduce the problem to consideration of $(k+1)$-point distributions. It would then be possible to write equations for $p_1, \ldots, p_{k+1}$ in terms of $y_1, \ldots, y_{k+1}$ and proceed as in Chapter 3. It is expected that all monotonicity results for replacement times as well as expected lifetimes would carry over from the known mean and variance model to the k-moment model. Further, once the single spare return function is found, it is not hard to find optimal schedules and expected lifetimes for multiple identical spares and this new type of spare could be integrated into the framework for optimal sequencing of different spares presented in Chapter 5.

There are several classes of distributions which yield uninteresting minimax schedules. One is the class of distributions with known mean. Since the exponential distribution is NWUE, its single spare return function is simply the maximum of its mean and the reward. Thus, the minimax single spare return function for spares with a given mean is also the maximum of the mean and the reward. In fact, even if we require that the spares have an IFR distribution, in addition to a given mean, we can do no better. This is because the exponential distribution is also IFR.

The class of distributions with more than one known percentile yields somewhat more interesting minimax schedules than the class with only one known percentile. These schedules are easy to find since there is one minimizing distribution, independent of the schedule. As in the single percentile model, the optimal schedules all involve replacement

times which are the known percentiles of the lifetime distribution. Thus, by determining the specific percentiles, the optimal schedule is, to a large extent, determined in advance.

New Models

The following three new models are useful in a practical sense, although they are also more complex analytically than models treated so far. The first is a model with deteriorating spares. By this we mean that the lifetime distribution of a spare is a function of its age upon installation. Thus, the shelf life of the spares is an important factor in the decision about when to make replacements.

Deteriorating Spares Model: Everything is the same as the Complete Information Model except that the lifetime distribution of a spare is a function of its age at the time it is installed. It is assumed that all spares begin with the same age. Thus, any spare installed at a particular time will have the same lifetime distribution as any other spare installed at that time.

Let F_t be the cumulative distribution function of a spare installed at time t. Spares which are deteriorating, but deteriorate more quickly when being used satisfy the following inequality:

$$\bar{F}_t(s) \geq \bar{F}_{t+\Delta}(s) \geq \bar{F}_t(s+\Delta), \quad t,s,\Delta \geq 0.$$

That is to say, a younger spare is better than an older spare and an unused spare is better than a used one. This apparently is the only

restriction on a family of lifetime distributions to be consistent with the idea of deteriorating spares.

It is simple to define a recursive relation yielding optimal schedules and expected lifetimes. Let $u(n,t)$ be the optimal expected lifetime beginning with n spares each of which has age t now. Then u satisfies the following recursion:

$$u(0,t) = 0, \; t \geq 0,$$

$$u(n+1,t) = \sup_{y \geq 0} \left\{ \int_0^y \bar{F}_t(x)dx + \bar{F}_t(y)u(n,t+y) \right\}, \; n \geq 0, \; t \geq 0.$$

The problem with solving this recursion lies in the fact that in order to find $u(n+1,t)$, we need to know $u(n,s)$ for all $s \geq t$. As in the Additional Spare Model, presented in Chapter 6, this makes numerical solution impractical. In fact, this model might be harder to solve numerically than the Additional Spare Model because, in the Additional Spare Model the reward function only had to be known for arguments in a finite interval.

It is straight-forward to prove that $u(n,t)$ is nondecreasing in n and nonincreasing in t. That is, it is better to have more spares and to have younger spares, as we should have suspected anyway. However, any other results seem difficult at best. The next step in the analysis of this model would be to further restrict the family of spare lifetime distributions, F_t. Presumably, the appropriate restriction would enable one to obtain more interesting results.

Another possibility is to apply the minimax approach to this problem. Once again, there is little hope for a good solution unless a closed form expression (in t) can be found for $u(n,t)$. However, it is probably true that the general results for the minimax case are the same as for the case of a known family of distributions.

For an even more difficult problem, the spares could initially have different ages, which would add the aspect of optimal sequencing to the problem of optimal scheduling. This extension of the Deteriorating Spares Model would still suffer from the same problems of the impracticality of numerical solution. This would be compounded by the necessity to check a number of different permutations.

Another interesting new model is one which allows spares to be reused after they have been replaced. Various assumptions are possible about what lifetime distribution a reused spare would have and how many times a spare could be reused.

The most obvious lifetime distribution assumption is that the lifetime distribution of a reused spare is simply the distribution of its remaining lifetime at the time it was removed. That is, there is no penalty for removing and then reinstalling a spare. If we also assume that spares can be reused infinitely often, then it is easy to see how a group of spares would be utilized. The installed spare would always be the one with the lowest current failure rate. If the installed spare's failure rate ever rose above the failure rate of the next "best" spare then they would be immediately exchanged. Note that this could lead to

continuous installation and removal of spares. This is obviously not a practical solution.

One way to deal with the problem of continuous swapping of spares would be to limit each spare to some fixed, finite number of reuses. Another way would be to significantly penalize a spare at the time it is removed so that it would not be optimal to install and remove a spare continuously. In either case, this problem becomes complicated quickly since once a number of spares are waiting to be reused, they will probably have different lifetime distributions. Therefore, an optimal sequence must be determined, taking into account that the spares can be reused again.

The final new model which will be discussed is actually a generalization of the Parallel Model presented in Chapter 6. Now we assume that spares may work in parallel, but that they can be installed and removed independently of each other.

<u>Super-Parallel Model</u>: This model is the same as the Complete Information Model, except that when a new spare is installed, the old spare(s) need not be removed. The system fails at the first time all installed spares have failed. This differs from the Parallel Model in that, with the Parallel Model, a group of spares must be removed before a new group may be installed.

The first question to be considered is whether the status of installed spares may be used in the new installation decisions. Presumably, the static model, in which all installation times must be

determined in advance, is easier to solve and the solution is easier to implement.

It is possible to define a recursion for an optimal static policy, although it seems difficult to solve. As in Chapter 2, we will take F to be the distribution function for the spares. Let $v(n,G)$ be the optimal expected lifetime with n uninstalled spares assuming that the currently installed group has the remaining lifetime distribution G. When no spares are available, then the only strategy is to wait for the installed group to fail.

$$v(0,G) = \int_0^\infty \bar{G}(x)dx, \text{ and}$$

$$v(n+1,G) = \sup_{t \geq 0} \left\{ \int_0^t \bar{G}(x)dx + v(n,H_t)\bar{G}(t) \right\},$$

where $$H_t(s) = \frac{G(t+s) - G(t)}{1-G(t)} F(s).$$

Thus, H_t is the remaining lifetime distribution of the newly expanded group, conditional on the old group surviving to t. Once again we find that the reward, $v(n,H_t)$, is a function of t. Thus, numerical solution is impractical and the recursion must be solved analytically.

We now consider the problem of determining an optimal dynamic policy: a policy which can take account of the fact that a certain installed spare has failed. An obvious strategy in this case is to initially install two spares and then install a new spare each time an

installed spare fails. It is not clear why such a policy would be optimal, but it has the nice property that, with probability 1, we use all of our spares. This is not the case with any other model, except for the parallel model in which we had the option of forming a single group containing all of the available spares.

One other possible variation of this model would be to limit the maximum number of spares which can be active at one time. In the Basic Model, this number is implicitly assumed to be one, but this assumption could be modified. For example, on a multiple engine aircraft, it is sensible to allow some finite number of spares greater than one to be active at one time.

Further exploration in these areas would have the effect of making the Basic Model more applicable to specific applications. Since the Basic Model, and especially the Minimax Model, were developed with the objective of being easy to implement, it is hoped that any extensions or variations would also pursue this objective.

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ABSTRACT: ON MAXIMIZING THE EXPECTED LIFETIME OF REPLACEABLE SYSTEMS
by Mark Mathiasen Perkins

Consider the following model. A system has one vital component with n spares. When the vital component fails, the system fails. Derman, Lieberman, and Ross have considered the problem of maximizing the time until failure of the system. They obtained optimal schedules when the lifetime distributions of the spares were known. This paper treats several different cases of this model and finds optimal schedules together with their properties.

Assuming only the first two moments of the spare component lifetime distributions are known, the minimax replacement schedule is obtained. These minimax replacement schedules are then compared with schedules based on different amounts of information.

When the spares are different from each other, it must be decided in which order they should be used. A general sufficient condition is given under which the greedy order is maximal. This condition applies when the complete lifetime distribution is known, or for any minimax schedule.

Two special cases are also considered. The first is the case in which groups of spares may be used in parallel. In the second special case, an additional spare will become available at some future time.

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